Supplementary Material: Photonic Quantum State Tomography Using Free Electrons

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Section I: Free-electron–photon interactions and comb electron states

In this section, we explore the fundamental properties of the interaction between free electrons and photons, focusing on its impact on electron comb states. The quantum mechanical interaction between a free electron and a single-mode photonic state is governed by the scattering matrix [37,38]:

$$\hat{S} = \exp\left[g_{\mathrm{q}}\hat{b}\hat{a}^\dagger - g_{\mathrm{q}}^*\hat{b}^\dagger\hat{a}\right], \tag{S1}$$

where $\hat{b}, \hat{b}^\dagger$ are the energy ladder operators for the free electron. The $\hat{b} \equiv e^{-i\frac{\omega}{v}z}$ ladder operator represents an electron losing energy $\hbar\omega$ under the paraxial and no-recoil approximations, with v being the electron velocity and z its longitudinal position operator. The b operator has the following commutation relation $\left[\hat{b}, \hat{b}^\dagger\right] = 0$.

The electron comb state is defined by the following equation:

$$|\mathrm{comb}(\phi)\rangle = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} e^{ik\phi}|E_k\rangle, \tag{S2}$$

where $|E_k\rangle$ is single energy electron state with energy $E_k = E_0 + k\hbar\omega$. Let us first demonstrate that the comb electron states are eigenstates of the scattering matrix Eq.(S1). By the definition of the comb electron, it is an eigenstate of the operator $\hat{b}$:

$$\hat{b}|\mathrm{comb}(\phi)\rangle = e^{i\phi}|\mathrm{comb}(\phi)\rangle. \tag{S3}$$

Substituting the comb state from Eq. (S2) to the scattering matrix Eq. (S1) and using Eq. (S3), we obtain:

$$\hat{S}|\mathrm{comb}(\phi)\rangle = \exp\left[g_{\mathrm{q}}e^{i\phi}\hat{a}^\dagger - g_{\mathrm{q}}^*e^{-i\phi}\hat{a}\right]|\mathrm{comb}(\phi)\rangle = \hat{D}\left(g_{\mathrm{q}}e^{i\phi}\right)|\mathrm{comb}(\phi)\rangle, \tag{S4}$$

where $\widehat{D}\left(g_{\mathrm{q}}e^{i\phi}\right)$ is the coherent shift operator acting on the photonic field.

Now, let's establish that $|\mathrm{comb}(\phi)\rangle$ forms a complete orthogonal basis for any arbitrary electron state. An arbitrary electron with discrete energies $|E_k\rangle$ can be expressed as:

$$|\psi_{\mathrm{e}}\rangle = \sum_k a_k |E_k\rangle\,, \tag{S5}$$

where $|E_k\rangle$ are single-energy states that form a complete orthogonal basis $\langle E_n|E_m\rangle = \delta_{nm}$. According to the definition of the comb electron, the single-energy state can be represented as:

$$|E_k\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{+\pi} e^{-ik\phi}|\mathrm{comb}(\phi)\rangle d\phi\,. \tag{S6}$$

Therefore, any electron state can be described as:

$$|\psi_{\mathrm{e}}\rangle = \int_{-\pi}^{+\pi} c_\phi |\mathrm{comb}(\phi)\rangle d\phi\,, \tag{S7}$$

where the coefficients c_ϕ and a_k are related by the Fourier series

$$\begin{cases} c_\phi = (2\pi)^{-\frac{1}{2}}\sum_k a_k e^{-ik\phi} \\ a_k = (2\pi)^{-\frac{1}{2}}\int_{-\pi}^{+\pi} c_\phi e^{ik\phi} d\phi \end{cases}. \tag{S8}$$

Lastly, we demonstrate the orthogonality of comb states:

$$\langle \mathrm{comb}(\phi_2)|\mathrm{comb}(\phi_1)\rangle = \frac{1}{2\pi}\sum_k e^{ik(\phi_1-\phi_2)} = \sum_{n=-\infty}^{+\infty} \delta(\phi_1-\phi_2-2\pi n)\,. \tag{S9}$$

This analysis reveals the intrinsic properties of free-electron–photon interactions and their effects on electron comb states, providing a foundation for further exploration of quantum aspects of the interaction.

Section II: Electron energy spectrum of an arbitrary electron state interacting with an arbitrary photonic state

In this section, we derive the electron energy spectrum of an arbitrary electron state after the interaction with an arbitrary quantum photonic state. This derivation shows the advantages of the comb electron basis, and we get the equation for the electron energy loss spectrum in the most general case.

We consider the most general initial state, which is a tensor product of an arbitrary pure electron state $|\psi_{\mathrm{e}}\rangle$ and an arbitrary photonic state ρ_{ph}, represented by a density matrix:

$$\rho = |\psi_{\mathrm{e}}\rangle\langle\psi_{\mathrm{e}}| \otimes \rho_{\mathrm{ph}}, \tag{S10}$$

where according to the previous section the electron state can be written in the comb basis $|\psi_{\mathrm{e}}\rangle = \int_{-\pi}^{+\pi} c_{\phi_1}|\mathrm{comb}(\phi_1)\rangle d\phi_1$. Consequently, the scattering matrix's action on the initial state is:

$$S\rho S^\dagger = \int_{-\pi}^{+\pi} d\phi_1 \int_{-\pi}^{+\pi} c_{\phi_1} c^*_{\phi_2} |\mathrm{comb}(\phi_1)\rangle\langle\mathrm{comb}(\phi_2)| \otimes \widehat{D}\left(g_\mathrm{q} e^{i\phi_1}\right)\rho_\mathrm{ph}\widehat{D}\left(-g_\mathrm{q} e^{i\phi_2}\right). \quad \text{(S11)}$$

The electron energy spectrum equals the probability of the electron having energy E_k after the interaction:

$$P_k = \mathrm{Tr_{ph}}\langle E_k|\rho|E_k\rangle =$$

$$= \int_{-\pi}^{+\pi} d\phi_1 \int_{-\pi}^{+\pi} d\phi_2\, c_{\phi_1} c^*_{\phi_2} \langle\mathrm{comb}(\phi_2)|E_k\rangle\langle E_k|\mathrm{comb}(\phi_1)\rangle \mathrm{Tr_{ph}}\left[\rho_\mathrm{ph}\widehat{D}\left(-g_\mathrm{q} e^{i\phi_2}\right)\widehat{D}\left(g_\mathrm{q} e^{i\phi_1}\right)\right].$$

Using the definition of the comb electron and the property of the coherent shift operators that $\widehat{D}(\alpha)\widehat{D}(\beta) = e^{(\alpha\beta^*-\alpha^*\beta)/2}\widehat{D}(\alpha+\beta)$ [2]:

$$P_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\phi_1 \int_{-\pi}^{+\pi} d\phi_2\, c_{\phi_1} c^*_{\phi_2} e^{ik(\phi_1-\phi_2)} e^{i|g_\mathrm{q}|^2 \sin(\phi_1-\phi_2)} C_s\left(g_\mathrm{q}\left(e^{i\phi_1}-e^{i\phi_2}\right)\right), \quad \text{(S12)}$$

where $C_s\left(g_\mathrm{q}\left(e^{i\phi_1}-e^{i\phi_2}\right)\right) = \mathrm{Tr_{ph}}\left[\rho_\mathrm{ph}\widehat{D}\left(g_\mathrm{q}\left(e^{i\phi_1}-e^{i\phi_2}\right)\right)\right]$ is the characteristic function of the photonic state. The characteristic function fully describes the quantum photonic state and is connected to with Wigner function by the Fourier transform [4]:

$$W(\beta) = \pi^{-2}\int d^2\xi\; C_\mathrm{s}(\xi) e^{\beta\xi^*-\beta^*\xi}. \quad \text{(S13)}$$

Now we can change variables $\phi \equiv \phi_1 - \phi_2, \alpha \equiv \phi_2$ and get:

$$P_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\phi e^{ik\phi}\, e^{i|g_\mathrm{q}|^2 \sin\phi} \int_{-\pi}^{+\pi} d\alpha\, c_{\phi+\alpha} c^*_\alpha C_s\left(g_\mathrm{q} e^{i\alpha}\left(e^{i\phi}-1\right)\right), \quad \text{(S14)}$$

We note that the electron energy spectrum Eq. (S14) is derived for the most general case of an arbitrary pure electron state and an arbitrary photonic density matrix. From Eq. (S14) it becomes evident that the characteristic function can only be reconstructed within the circle of radius $2|g_\mathrm{q}|$. This sets the fundamental limit on how much information we can reconstruct from the electron energy spectrum.

Section III: The reconstruction of photonic state using electron energy spectrum

In the previous section, we showed that the spectrum contains information about the characteristic function of the photonic state within the circle of radius $2|g_\mathrm{q}|$. Let's now show that the characteristic function $C_s(\xi)$ cannot be reconstructed from unshaped electrons and can be reconstructed using electron homodyne detection.

Firstly, Eq. (S14) can be further simplified. We can first find the Fourier series of the electron energy spectrum P_k:

$$P(\phi) \equiv e^{-i|g_\mathrm{q}|^2 \sin\phi} \sum_{k=-\infty}^{+\infty} P_k e^{-ik\phi}. \quad \text{(S15)}$$

Indeed, if we know P_k we can easily find $P(\phi)$ according to Eq. (S15). Furthermore, we easily can reverse the equation and find P_k if we know $P(\phi)$:

$$P_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\phi\, e^{ik\phi} e^{i|g_\mathrm{q}|^2 \sin\phi} P(\phi). \tag{S16}$$

The advantage of $P(\phi)$ is that it is directly connected with characteristic function. According to Eq. (S14), $P(\phi)$ is connected with the characteristic function by the following relation:

$$P(\phi) = \int_{-\pi}^{+\pi} d\alpha\, c_{\phi+\alpha} c_\alpha^* C_s\left(g_\mathrm{q} e^{i\alpha}\left(e^{i\phi} - 1\right)\right). \tag{S17}$$

Now let us see what electron state with coefficients c_ϕ allow reconstruction of C_s.

<u>Unshaped electrons:</u>

Firstly, let us consider single-energy electron $|E_0\rangle$ with energy E_0. Its Fourier transform gives:

$$c_\phi = (2\pi)^{-1/2}. \tag{S18}$$

Then accordingly we have:

$$P(\phi) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\alpha\, C_s\left(g_\mathrm{q} e^{i\alpha}\left(e^{i\phi} - 1\right)\right) \tag{S19}$$

Indeed, Eq. (S19) is not invertible. We can only get $C_s(\xi)$ averaged over all possible phases. Eq. (19) shows that the electron spectrum of unshaped electrons cannot reconstruct a quantum photonic state.

<u>Electron homodyne detection:</u>

Let us now show that the electron shaped by a coherent classical photonic state phase-locked with the target quantum state can reconstruct this quantum state. The electron state after the interaction with the coherent photonic state is described by [37]:

$$|\psi_\mathrm{e}\rangle = \sum_{k=-\infty}^{+\infty} J_k(|\beta|) e^{ik\theta}\, |E_k\rangle, \tag{S20}$$

where $|\beta|e^{i\theta}$ is the semi-classical coupling of a coherent state $|\alpha\rangle$ with free electrons. This coupling is connected to g_q by the following equation: $|\beta|e^{i\theta} = g_\mathrm{q}\alpha$. The coefficients in the comb basis according to Eq. (S8) are equal to:

$$c_\phi = (2\pi)^{-\frac{1}{2}} \sum_k a_k e^{-ik\phi},$$

where according to Eq. (S20), the coefficients in the energy basis $\sum_k a_k |E_k\rangle$ equal to $a_k = J_k(|\beta|)e^{ik\theta}$. Then, the coefficients equal:

$$c_\phi = (2\pi)^{-\frac{1}{2}} \sum_k J_k(|\beta|) e^{ik(\theta-\phi)}.$$

We use the Laurent series $e^{x\cos\phi} = \sum_k J_k(x) e^{i\phi k}$, and get:

$$c_\phi(\theta) = (2\pi)^{-1/2} e^{2|\beta| i \sin(\theta-\phi)}. \tag{S21}$$

Then, according to Eq. (S17), we get:

$$P(\phi;\theta) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\alpha\, e^{2|\beta|i(\sin(\theta-\phi-\alpha)-\sin(\theta-\alpha))}\left(g_{\mathrm{q}}e^{i\alpha}\left(e^{i\phi}-1\right)\right). \tag{S22}$$

We change variables $\alpha \to \alpha + \phi/2$ and get:

$$P(\phi;\theta) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\alpha\, e^{-4|\beta|i\sin\phi\cos(\alpha-\theta)} C_{\mathrm{s}}\left(2ig_{\mathrm{q}}e^{i\alpha}\sin\phi/2\right). \tag{S23}$$

We use once again the Laurent series $e^{z\frac{(w+w^{-1})}{2}} = \sum_{n=-\infty}^{\infty} J_n(z)w^n$ and get:

$$e^{-4|\beta|i\sin\phi\cos(\alpha-\theta)} = \sum_{n=-\infty}^{+\infty} J_n(4|\beta|\sin\phi)i^n\, e^{in\alpha}e^{-in\theta}\,. \tag{S24}$$

Then, Eq. (S23) can be written as:

$$P(\phi;\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{-in\theta}\left(J_n(4|\beta|\sin\phi)i^n\int_{-\pi}^{+\pi} d\alpha\, e^{in\alpha} C_{\mathrm{s}}\left(2ig_{\mathrm{q}}e^{i\alpha}\sin\frac{\phi}{2}\right)\right). \tag{S25}$$

Now, using the Fourier series we can find $P_n(\phi) = \frac{1}{\sqrt{2\pi}}\frac{1}{J_n(4|\beta|\sin\phi)\, i^n}\int_{-\pi}^{+\pi} d\theta e^{in\theta}P(\phi;\theta)$:

$$P_n(\phi) = \int_{-\pi}^{+\pi} d\alpha\, e^{in\alpha}\, C_s\left(2ig_q e^{i\alpha}\sin\phi/2\right). \tag{S26}$$

Applying the Fourier series once again, we get:

$$C_s\left(2ig_q e^{i\alpha}\sin\phi/2\right) = \sum_{n=-\infty}^{\infty} P_n(\phi)e^{-in\alpha}\,. \tag{S27}$$

Combining all the equations, we get:

$$C_s\left(2ig_q e^{i\alpha}\sin\frac{\phi}{2}\right) = \sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{e^{-in\alpha}}{J_n(4|\beta|\sin\phi)\, i^n}\int_{-\pi}^{+\pi} d\theta e^{in\theta}P(\phi;\theta) \tag{S28}$$

If we substitute the definition of $P(\phi,\theta)$, we will get the connection between the characteristic function and the electron energy spectrum:

$$C_s\left(2ig_q e^{i\alpha}\sin\frac{\phi}{2}\right) = \frac{e^{-i|g_q|^2\sin\phi}}{\sqrt{2\pi}}\sum_n \frac{e^{-in\alpha}}{J_n(4|\beta|\sin\phi)\, i^n}\int_{-\pi}^{+\pi} d\theta\, e^{in\theta}\sum_k P_k(\theta)e^{-ik\phi}\,, \tag{S29}$$

Importantly, in the case of $|\beta| \gg 1$ the reconstruction can be done even in a simpler way. Eq. (S23) in this case can be approximated with stationary phase approximation:

$$P(\phi;\theta) \approx (8\pi|\beta||\sin\phi|)^{-1/2}e^{-4|\beta||\sin\phi|i+\frac{i\pi}{4}}C_s\left(2ig_q e^{i\theta}|\sin\phi/2|\right) +$$

$$+(8\pi|\beta||\sin\phi|)^{-1/2}e^{4|\beta||\sin\phi|i-\frac{i\pi}{4}}C_s\left(-2ig_q e^{i\theta}|\sin\phi/2|\right). \tag{S30}$$

This allows us to find $C_s\left(2ig_q e^{i\theta}|\sin\phi/2|\right)$ without any integrals:

$$C_s\left(2ig_q e^{i\theta}\left|\sin\frac{\phi}{2}\right|\right) = \frac{\sqrt{8\pi|\beta||\sin\phi|}}{2i\cos(8|\beta||\sin\phi|)}\left(e^{-4|\beta||\sin\phi|i+\frac{i\pi}{4}}P(\phi,\theta) - e^{4|\beta||\sin\phi|i-\frac{i\pi}{4}}P(\phi,\theta+\pi)\right). \tag{S31}$$

Eq (S28) and Eq. (S31) allow us to reconstruct the characteristic function in the circle of radius $2|g_{\mathrm{q}}|$, thus extracting the maximal possible information from the spectrum.

Section IV: The overlap of the reconstruction depending on the interaction constant

In the preceding section, we demonstrated the feasibility of reconstructing the characteristic function of a photonic state within a circle of radius $2|g_{\mathrm{q}}|$. Let's proceed under the assumption that our knowledge of the characteristic function is limited to this circle, and beyond this boundary—specifically outside the circle of radius $2|g_{\mathrm{q}}|$—the characteristic function is considered to be zero. Under these conditions, we aim to compute the overlap between the reconstructed state and the actual state as a function of the magnitude of the coupling constant, $|g_{\mathrm{q}}|$. The overlap between the measured characteristic function, $C_{\mathrm{measured}}(\xi)$, and the actual characteristic function, $C(\xi)$, of the state is defined by the equation:

$$\mathrm{overlap} = \frac{1}{\int |C(\xi)|^2 d^2\xi} \left| \iint C_{\mathrm{measured}}(\xi) C^*(\xi) d^2\xi \right|. \tag{S32}$$

From Eq. (S32), it is clear that the overlap can achieve a maximum value of 1, indicating perfect overlap between the measured and actual states, and a minimum value of 0, signifying complete dissimilarity. In the realm of quantum photonic states, this overlap metric serves as an indicator of the phase space similarity between two states. This concept is closely related to fidelity, where both metrics are designed to quantify the degree of similarity, or "closeness," between two quantum states. Notably, fidelity and overlap share an identical value range: 0 for states that are distinctly different, and 1 for identical states. To elucidate this relationship further, we plot the overlap of the measured characteristic function with the characteristic function of the quantum signal as a function of the coupling constant $|g_{\mathrm{q}}|$, as illustrated in Fig. S1. This visualization reveals a direct correlation: as $|g_{\mathrm{q}}|$ increases, so does the overlap, thereby enhancing the precision of the reconstruction.

Additionally, we include the table detailing the characteristic and Wigner functions of the photonic states under consideration. This table serves as a reference for understanding the diverse behaviors and properties of various photonic states within the specified framework.

Table S1: Fock representation, Wigner, and characteristic functions of the photonic states

Name	Fock basis	Wigner function $W(\beta)$	Characteristic function $C(\xi)$
Coherent	$\lvert\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \lvert n\rangle$	$\frac{2}{\pi} e^{-2\lvert\beta-\alpha\rvert^2}$	$e^{-\lvert\xi\rvert^2} e^{\xi\alpha^* - \xi^*\alpha}$
Fock	$\lvert n\rangle$	$\frac{2(-1)^n}{\pi} e^{-\lvert\beta\rvert^2} L_n(2\lvert\beta\rvert^2)$	$e^{-\frac{1}{2}\lvert\xi\rvert^2} L_n(\lvert\xi\rvert^2)$
Cat	$\lvert\mathrm{cat}\rangle = \frac{\lvert\alpha\rangle \pm \lvert-\alpha\rangle}{\sqrt{N_\pm}}$, $N_\pm = 2(1 \pm e^{-2\lvert\alpha\rvert^2})$	$\frac{2}{\pi N_\pm} e^{-2\lvert\beta-\alpha\rvert^2} + e^{-2\lvert\beta+\alpha\rvert^2}$ $\pm 2e^{-\lvert\beta\rvert^2} \cos(4\alpha \mathrm{Im}[\beta])$	$2e^{-\frac{1}{2}\lvert\xi\rvert^2} \cos(\xi\alpha^* - \xi^*\alpha) +$ $2e^{-\frac{1}{2}\lvert\xi\rvert^2 - 2\lvert\alpha\rvert^2} \cos(\xi\alpha^* + \xi^*\alpha)$
Mixed Cat	$\rho_{\mathrm{mixed}} = \frac{\lvert\alpha\rangle + \langle-\alpha\rvert}{2}$	$\frac{1}{\pi}\left(e^{-2\lvert\beta-\alpha\rvert^2} + e^{-2\lvert\beta+\alpha\rvert^2}\right)$	$e^{-\frac{1}{2}\lvert\xi\rvert^2} \cos(\xi\alpha^* - \xi^*\alpha)$
Squeezed	$\sum_{m=0}^{\infty} \frac{(-1)^m \sqrt{(2m)!}\, e^{im\theta} (\tanh r)^m}{\sqrt{\cosh r}\, 2^m m!} \lvert 2m\rangle$	$\frac{1}{\pi} e^{-\lvert\cosh r e^{-i\theta}\beta + \sinh r e^{i\theta}\beta^*\rvert^2}$	$e^{-\frac{1}{2}\lvert\cosh r e^{-i\theta}\xi^* + \sinh r e^{i\theta}\xi\rvert^2}$
Thermal	$\rho_{\mathrm{th}} = \sum_{n=0}^{\infty} \frac{1}{\langle n\rangle + 1} \left(\frac{\langle n\rangle}{\langle n\rangle + 1}\right)^n \lvert n\rangle\langle n\rvert$	$\frac{2e^{-\frac{\lvert\beta\rvert^2}{2\langle n\rangle+1}}}{\pi(2\langle n\rangle + 1)}$	$e^{-\frac{2\langle n\rangle+1}{2}\lvert\xi\rvert^2}$

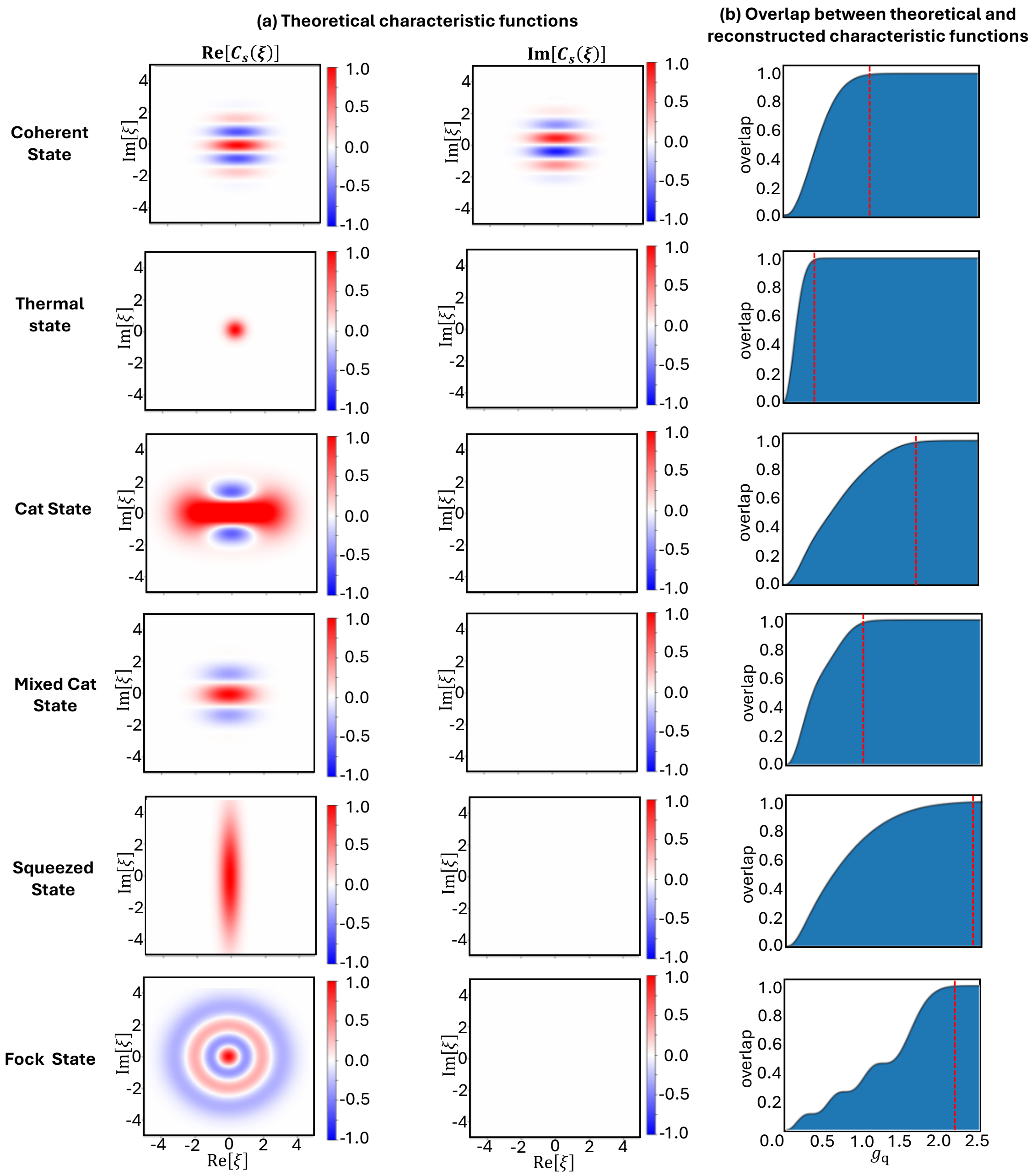

Fig S1: The overlap between the quantum signal and the reconstructed signal. The reconstruction is based on the conservative choice of a zero characteristic function outside the circle of radius $2|g_\mathrm{q}|$. **(a)** Theoretical (i.e., expected) characteristic functions for coherent, thermal, cat, mixed cat, squeezed vacuum, and Fock states. The first row illustrates the real parts, while the second row illustrates the imaginary parts of the characteristic functions. **(b)** The overlap between the theoretical and the reconstructed characteristic functions, defined by Eq. (S32), as a function of the strength of free-electron-photon coupling g_q. The red line indicates when the overlap is 0.95. The parameters are taken the same as in the main text. In Fig. 3, the mean photon number is 3 for the Fock, coherent, thermal, and squeezed vacuum states, or 1 for the cat and mixed-cat states.