Parametric amplification of a quantum pulse

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CONTENTS

I. MODES OF OUTPUT FIELD

We present here the derivation of the first-order coherence function $g_1(\omega_1, \omega_2)$ and its decomposition into modes. The Bogoliubov transformation explained in the main text yields the correlation function:

$$\begin{aligned} g_1(\omega_1,\omega_2) = &\int d\omega' d\omega'' F^*(\omega_1,\omega')F(\omega_2,\omega'')\,\langle a_{\rm in}^\dagger(\omega')a_{\rm in}(\omega'')\rangle + \int d\omega' d\omega'' F^*(\omega_1,\omega')G^*(\omega_2,\omega'')\,\langle a_{\rm in}^\dagger(\omega')a_{\rm in}^\dagger(\omega'')\rangle \\ &+ \int d\omega' d\omega'' G(\omega_1,\omega')F(\omega_2,\omega'')\,\langle a_{\rm in}(\omega')a_{\rm in}(\omega'')\rangle + \int d\omega' d\omega'' G(\omega_1,\omega')G^*(\omega_2,\omega'')\,\langle a_{\rm in}^\dagger(\omega'')a_{\rm in}(\omega')\rangle \\ &+ \int d\omega' G(\omega_1,\omega')G^*(\omega_2,\omega'). \end{aligned} \tag{1}$$

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For the special case where the input is vacuum except for a single mode with a spectral (temporal) profile $u(\omega)(u(t))$, and a quantum state density matrix ρ_u, we can factor the correlation functions as

$$\langle a^\dagger_{\rm in}(\omega')a_{\rm in}(\omega'')\rangle = \langle a^\dagger_u a_u\rangle u^*(\omega')u(\omega''), \tag{2}$$

and similarly for the expectation values of the form $\langle a_{\rm in}(\omega')a_{\rm in}(\omega'')\rangle = \langle a_u a_u\rangle u(\omega')u(\omega'')\rangle$. We can thus rewrite the expression for g_1 using Eq. (2):

$$\begin{aligned} g_1(\omega_1,\omega_2) = &\langle a^\dagger_u a_u\rangle \int d\omega' F^*(\omega_1,\omega')u^*(\omega') \int d\omega'' F(\omega_2,\omega'')u(\omega'') + \langle a^\dagger_u a^\dagger_u\rangle \int d\omega' u^*(\omega') F^*(\omega_1,\omega') \int u^*(\omega'') G^*(\omega_2,\omega'') \\ &+ \langle a_u a_u\rangle \int d\omega' G(\omega_1,\omega')u(\omega') \int F(\omega_2,\omega'')u(\omega'')d\omega'' + \langle a^\dagger_u a_u\rangle \int d\omega' G(\omega_1,\omega')u(\omega') \int d\omega'' G^*(\omega_2,\omega'')u^*(\omega'') \\ &+ \int d\omega' G(\omega_1,\omega')G^*(\omega_2,\omega'). \end{aligned} \tag{3}$$

A. Proof that the input mode feeds only into two modes of the output field

To simplify the notation, we denote $\langle a^\dagger_u a_u\rangle = \alpha$ and $\langle a_u a_u\rangle = \beta$, where α is real, and β can be complex. We write the part of the g_1 function that depends on the input state, excluding the vacuum contribution, in matrix form as

$$g_1^{\rm dep} = \alpha f_u f_u^\dagger + \beta^* g_u f_u^\dagger + \beta f_u g_u^\dagger + \alpha g_u g_u^\dagger. \tag{4}$$

where $f_u(f_u^\dagger)$ and $g_u(g_u^\dagger)$ are column(row) vectors, representing the frequency dependence, $f_u(\omega_1) = \int d\omega' F(\omega_1,\omega')u(\omega')$ and $g_u(\omega_1) = \int d\omega' G^*(\omega_1,\omega')u^*(\omega')$. We denote the length of f_u as a and the length of g_u as b, and rewrite $f_u(\omega) = a\tilde{f}_u(\omega)$ and $g_u(\omega) = b\tilde{g}_u(\omega)$, where $\tilde{f}_u$ and $\tilde{g}_u$ are normalized. We now expand $\tilde{g}_u$ into a component parallel to $\tilde{f}_u$ and an orthogonal function $\tilde{h}_u$, $\tilde{g}_u = (\gamma\tilde{f}_u + \delta\tilde{h}_u)$. This leads to

$$g_1^{\rm dep} = \alpha|a|^2\tilde{f}_u\tilde{f}_u^\dagger + \beta^* a^* b(\gamma\tilde{f}_u + \delta\tilde{h}_u)\tilde{f}_u^\dagger + \beta a b^*\tilde{f}_u(\gamma\tilde{f}_u + \delta\tilde{h}_u)^\dagger + \alpha|b|^2(\gamma\tilde{f}_u + \delta\tilde{h}_u)(\gamma\tilde{f}_u + \delta\tilde{h}_u)^\dagger. \tag{5}$$

Now we can write g_1 as a 2x2 matrix in the basis of these two function, which guarantees a maximum of 2 non-zero eigenvalues, and thus a maximum of two modes that are input dependent.

B. Population of modes

Having established that at most two modes in the output field depend on the input quantum state, the natural next question is, what is the effect of the initial state and spectral (temporal) mode of the pulse on the population of these two output modes. In the following, we analyse both of these dependencies. To do so, we use the decomposition of g_1 as a 2x2 matrix (see previous section for definitions), which yields

$$g_1 = \begin{pmatrix} \alpha|a|^2 + \beta^* a^* b\gamma + \beta a b^*\gamma^* + \alpha|b|^2|\gamma|^2 & \beta^* a^* b\delta + \alpha|b|^2\gamma^*\delta \\ \beta a b^*\delta^* + \alpha|b|^2\gamma\delta^* & \alpha|b|^2|\delta|^2 \end{pmatrix}. \tag{6}$$

This matrix will in general have two non-vanishing eigenvalues, while a vanishing eigenvalue implies vanishing of its determinant given by

$$|a|^2|b|^2|\delta|^2\left(\alpha^2 - |\beta|^2\right) = 0 \tag{7}$$

which is fulfilled if $a = 0$, $b = 0$ or $\delta = 0$, all corresponding to the g_1-function being single mode (a vanishing overlap of u with either F or G or that $f_u = g_u$). Note that the case $a = 0$ or $b = 0$ corresponds to the trivial case, where no amplification is achieved, and the transformation reduces to a linear transformation of the wave packet mode. A more interesting case leading to a vanishing determinant concerns the input quantum state,

$$\alpha = |\beta| \Leftrightarrow \langle a^\dagger_u a_u\rangle = |\langle a_u a_u\rangle|. \tag{8}$$

It is interesting that this simple property of the quantum state of the input pulse also ensures that it will only populate a single output mode.

C. Examples of input quantum states

If the input is in a coherent state $\rho_u = |\alpha\rangle\langle\alpha|$ in a tensor product with vacuum in all orthogonal modes, the expectation values factor, for example $\langle\alpha|_u\, a_{\rm in}(\omega')a_{\rm in}(\omega'')\,|\alpha\rangle_u = \alpha^2 u(\omega')u(\omega'')$. Inserting this gives the g_1-function

$$\begin{aligned}
g_1^{(\text{coherent})}(\omega_1,\omega_2) &= |\alpha|^2 \int d\omega' F^*(\omega_1,\omega')u^*(\omega') \int d\omega'' F(\omega_2,\omega'')u(\omega'') + (\alpha^*)^2 \int d\omega' F^*(\omega_1,\omega')u^*(\omega') \int d\omega'' G^*(\omega_2,\omega'')u^*(\omega'') \\
&+ \alpha^2 \int d\omega' G(\omega_1,\omega')u(\omega') \int d\omega'' F(\omega_2,\omega'')u(\omega'') + |\alpha|^2 \int d\omega' G(\omega_1,\omega')u(\omega') \int d\omega'' G^*(\omega_2,\omega'')u^*(\omega'') \\
&+ \int d\omega' G^*(\omega_1,\omega')G(\omega_2,\omega') \\
&= (\alpha^* f_u^*(\omega_1) + \alpha g_u^*(\omega_1))(\alpha f_u(\omega_2) + \alpha^* g_u(\omega_2)) + \int d\omega' G^*(\omega_1,\omega')G(\omega_2,\omega'),
\end{aligned} \tag{9}$$

where the first term is of the form $v_1(\omega_1)v_1^*(\omega_2)$ and the second term is independent of the input quantum state: the information of the input state is contained in a single output mode.

If, on the other hand, the input is in a Fock state, $\rho_u = |n\rangle\langle n|$, the input will seed not only a single mode, but instead two modes as

$$\begin{aligned}
g_1^{(\text{Fock})}(\omega_1,\omega_2) &= n \int d\omega' F^*(\omega_1,\omega')u^*(\omega') \int d\omega'' F(\omega_2,\omega'')u(\omega'') + n \int d\omega' G(\omega_1,\omega')u(\omega') \int d\omega'' G^*(\omega_2,\omega'')u^*(\omega'') \\
&+ \int d\omega' G(\omega_1,\omega')G^*(\omega_2,\omega') = n f_u^*(\omega_1) f_u(\omega_2) + n g_u^*(\omega_1) g_u(\omega_2) + \int d\omega' G^*(\omega_1,\omega')G(\omega_2,\omega').
\end{aligned} \tag{10}$$

Note that $f_u(\omega)$ and $g_u(\omega)$ are in general not orthogonal, and we will have to apply a Gram-Schmidt decomposition to properly identify the quantum states populating two orthogonal modes.

Lets now consider also a squeezed vacuum input state given by: $|\psi\rangle = S(re^{i\theta})\,|0\rangle$ in the single mode $u(t)$. We can calculate the correlators $\langle a_u^\dagger a_u\rangle = \sinh^2(r)$, $\langle a_u a_u\rangle = e^{i\theta}\sinh(r)\cosh r$, and we immediately notice that $\langle a_u^\dagger a_u\rangle \neq |\langle a_u a_u\rangle|$ and thus the input squeezed state will split into two modes on the output. This clarifies that the multimode considerations presented in this work are important even for Gaussian input states.

II. QUANTUM STATE OF THE OUTPUT FIELD

Here we give more details on the procedure for extracting the quantum state in a given mode $v(\omega)$. Starting with the Bogoliubov transformation of equation (4) in the main text, we use $a_{v,\rm out} = \int_{-\infty}^{\infty} d\omega v^*(\omega)\, a_{\rm out}(\omega)$, and insert the expression for $a_{\rm out}$,

$$a_{v,\rm out} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega d\omega' \left(v^*(\omega)\,F(\omega,\omega')\,a_{\rm in}(\omega') + v^*(\omega)\,G^*(\omega,\omega')\,a_{\rm in}^\dagger(\omega')\right). \tag{11}$$

Now, defining $f(\omega) = \int_{-\infty}^{\infty} d\omega' v(\omega)\,F^*(\omega,\omega')$ and $g(\omega) = \int_{-\infty}^{\infty} d\omega' v^*(\omega)\,G^*(\omega,\omega')$ we can write:

$$a_{v,\rm out} = \zeta \int_{-\infty}^{\infty} d\omega' f^*(\omega') a_{\rm in}(\omega') + \xi \int_{-\infty}^{\infty} d\omega' g(\omega) a_{\rm in}^\dagger(\omega') \equiv \zeta a_f + \xi a_g^\dagger. \tag{12}$$

Next, as described in the main text, we must expand the functions f, g along the input u and along orthogonal modes that are also orthogonal to u. This results in the mode

$$\begin{aligned}
a_{v,\rm out} &= \zeta\,\langle f,u\rangle\,a_u + \xi\,\langle u,g\rangle\,a_u^\dagger \\
&+ \zeta\sqrt{1-|\langle f,u\rangle|^2}\,\langle h,k\rangle\,a_k + \xi\sqrt{1-|\langle u,g\rangle|^2}a_k^\dagger \\
&+ \zeta\sqrt{1-|\langle f,u\rangle|^2}\sqrt{1-|\langle k,h\rangle|^2}a_s,
\end{aligned} \tag{13}$$

with mode functions given by:

$$
\begin{aligned}
k(\omega) &= \frac{g(\omega) - u(\omega)\langle u, g\rangle}{\sqrt{1 - |\langle u, g\rangle|^2}} \\
h(\omega) &= \frac{f(\omega) - u(\omega)\langle u, f\rangle}{\sqrt{1 - |\langle u, f\rangle|^2}} \\
s(\omega) &= \frac{h(\omega) - k(\omega)\langle k, h\rangle}{\sqrt{1 - |\langle k, h\rangle|^2}}.
\end{aligned} \tag{14}
$$

A. Bloch Messiah Reduction

In the main article, we arrived at the three mode Bogoliubov transformation, of the form

$$\tilde{A}a_u + \tilde{B}a_u^\dagger + \tilde{C}a_k + \tilde{D}a_k^\dagger + \tilde{E}a_s \tag{15}$$

where u, k, s are orthogonal modes. First we can decompose this into:

$$\cos(\varphi)\left(Aa_u + Ba_u^\dagger + Ca_k + Da_k^\dagger\right) + \sin(\varphi)\, a_s \tag{16}$$

with $\cos(\varphi) = \sqrt{\left|\tilde{A}\right|^2 - \left|\tilde{B}\right|^2 + \left|\tilde{C}\right|^2 - \left|\tilde{D}\right|^2}$ and $\sin(\varphi)\, e^{i\phi} = \tilde{E}$. This allows now to look into a simpler two mode Bogoliubov transformation of the form:

$$a_{out} = Aa_u + Ba_u^\dagger + Ca_k + Da_k^\dagger \tag{17}$$

and we can also find that to fulfil the commutation relations, $\left[a_{\text{out}}, a_{\text{out}}^\dagger\right] = 1$, we must have $|A|^2 - |B|^2 + |C|^2 - |D|^2 = 1$.

To use the Bloch-Messiah decomposition, we need to find an orthogonal mode to a_{out}. In the case of real coefficients (corresponding to real pulse modes), the orthogonal mode is easily found to be

$$b_{\text{out}} = -Ca_u + Da_u^\dagger + Aa_k - Ba_k^\dagger. \tag{18}$$

The case of complex coefficients are a bit more complicated. To be an orthogonal mode, the b_{out} mode should obey

$$[b_{\text{out}}, a_{\text{out}}] = 0 \tag{19}$$

$$[b_{\text{out}}, a_{\text{out}}^\dagger] = 0 \tag{20}$$

$$[b_{\text{out}}, b_{\text{out}}^\dagger] = 1 \tag{21}$$

Defining the b_{out} mode as

$$b_{\text{out}} = A_2 a_u + B_2 a_u^\dagger + C_2 a_k + D_2 a_k^\dagger \tag{22}$$

the first two equations can be solved to give the solution

$$A_2 = \frac{C_2(DB^* - AC^*) + D_2(AD^* - CB^*)}{|A|^2 - |B|^2} \tag{23}$$

$$B_2 = \frac{C_2(DA^* - BC^*) + D_2(BD^* - CA^*)}{|A|^2 - |B|^2} \tag{24}$$

where C_2 and D_2 can be chosen arbitrarily. This choice should be made such that the mode can be normalized to obey $[b_{\text{out}}, b_{\text{out}}^\dagger] = 1$.

Using the Bloch Messiah reduction [1, 2] (assuming real coefficients) we can rewrite an arbitrary two mode Bogoliubov transformation as a two port beam splitter followed by two single mode squeezers and another beam splitter, see supplementary information of ref [1] for a repeat of the proof. Specifically,

$$\begin{pmatrix} a_{out} \\ b_{out} \end{pmatrix} = \underbrace{\begin{pmatrix} A & C \\ -C & A \end{pmatrix}}_{L_1} \begin{pmatrix} a_u \\ a_k \end{pmatrix} + \underbrace{\begin{pmatrix} B & D \\ D & -B \end{pmatrix}}_{L_2} \begin{pmatrix} a_u^\dagger \\ a_k^\dagger \end{pmatrix} \tag{25}$$

the Bloch messiah reduction is the statement that there exist unitary matrices V, W and diagonal matrices Σ_1, Σ_2 such that:

$$
\begin{aligned}
L_1 &= V\Sigma_1 W^T, \\
L_2 &= V\Sigma_2 W^\dagger.
\end{aligned}
$$

Then, the output quantum state can be found, using beam-splitter and single mode squeezing operations according to the form

$$
|\psi_f\rangle = U_{BS}\left(\theta_2, \phi_2\right) S_1\left(r_1\right) S_2\left(r_2\right) U_{BS}\left(\theta_1, \phi_1\right) |\psi_u\rangle |0\rangle\,, \tag{26}
$$

where θ_1, ϕ_1 are found from the rotation parameters of W and θ_2, ϕ_2 are found from the rotation parameters of V. Next, we would like to find V, W. First, $L_{1(2)}$ have a singular value decomposition, such that

$$
\begin{aligned}
L_1 &= \Sigma_1 \mathcal{U}_1^\dagger, \\
\Sigma_1 &= \begin{pmatrix} \sqrt{A^2 + C^2} & 0 \\ 0 & \sqrt{A^2 + C^2} \end{pmatrix}, \\
\mathcal{U}_1 &= \frac{1}{\sqrt{|A|^2 + |C|^2}} \begin{pmatrix} A & -C \\ C & A \end{pmatrix}.
\end{aligned}
$$

and

$$
\begin{aligned}
L_2 &= \Sigma_2 \mathcal{U}_2^\dagger, \\
\Sigma_2 &= \begin{pmatrix} \sqrt{|B|^2 + |D|^2} & \\ & \sqrt{|B|^2 + |D|^2} \end{pmatrix}, \\
\mathcal{U}_2 &= \frac{1}{\sqrt{|B|^2 + |D|^2}} \begin{pmatrix} B & D \\ D & -B \end{pmatrix}.
\end{aligned}
$$

Note that $\mathcal{U}_1^* \neq \mathcal{U}_2$ and so we must introduce a balancing matrix E. This matrix should obey $\mathcal{U}_1^\dagger \mathcal{U}_2^* = EE^T$, and then we will have that

$$
L_1 = E\Sigma_1 \left(E\mathcal{U}_2\right)^T \tag{27}
$$

and

$$
L_2 = E\Sigma_2 \left(E\mathcal{U}_2\right)^\dagger \tag{28}
$$

in the Bloch Messiah form that we need. We now need to find E.

$$
EE^T = \mathcal{U}_1^\dagger \mathcal{U}_2^* = \frac{1}{\sqrt{|A|^2 + |C|^2}} \frac{1}{\sqrt{|B|^2 + |D|^2}} \begin{pmatrix} A & C \\ -C & A \end{pmatrix} \begin{pmatrix} B & D \\ D & -B \end{pmatrix} \tag{29}
$$

$$
= \frac{1}{\sqrt{|A|^2 + |C|^2}} \frac{1}{\sqrt{|B|^2 + |D|^2}} \begin{pmatrix} AB + CD & AD - BC \\ AD - BC & -(AB + CD) \end{pmatrix} \tag{30}
$$

now it is possible to pick E such that $E = E^T$, giving $E^2 = \mathcal{U}_1^\dagger \mathcal{U}_2^*$ where E is found numerically as the square root of the matrix $\mathcal{U}_1^\dagger \mathcal{U}_2^*$. Having found E numerically, the Bloch-Messiah reduction is concluded, and the output state can be found in an equivalent Schrödinger picture.

The case of complex coefficients are a bit more complicated. In this case, the Bloch-Messiah decomposition can be performed numerically, using the provided Python code [3].

B. Schrodinger Picture

The final step into the solution for the output quantum state in a specified mode involves extracting the parameters for the Schrodinger picture transformation, which are θ_i, ϕ_i and r_i in equation (10) in the main text. Given a transformation:

$$U_{bs}(\theta,\phi) = \exp\left(i\theta\left(a^\dagger b e^{i\phi} + a b^\dagger e^{-i\phi}\right)\right), \tag{31}$$

we can find the transformation of the operators $a_{\text{out}}, b_{\text{out}}$:

$$\begin{aligned} a_{\text{out}} &= \cos(\theta)\, a_{\text{in}} + i e^{i\phi} \sin(\theta)\, b_{\text{in}} \\ b_{\text{out}} &= i e^{-i\phi} \sin(\theta)\, a_{\text{in}} + \cos(\theta)\, b_{\text{in}} \end{aligned} \tag{32}$$

The squeezing operator is given by

$$S(r) = \exp\left(\frac{1}{2}\left(r\left(a^\dagger\right)^2 - r^* a^2\right)\right), \tag{33}$$

which results in the single-mode squeezing operation

$$a_{\text{out}} = a_{\text{in}} \cosh|r| + a_{\text{in}}^\dagger \sinh|r| e^{i\lambda}, \tag{34}$$

where $r = |r|e^{i\lambda}$. The parameters θ_i, ϕ_i and r_i are found by comparison to the matrices of the transformation in the previous section.

C. Two mode quantum state

The previous sections described how to find the output quantum state of a single mode of the output field. As the output can only be in a maximum of 2 output states, it is natural to extend this to the case of finding the combined state of two output modes. The generalization is quite straightforward. First the decomposition into modes is performed for both modes as in the single mode case to achieve

$$a_{v_1} = \tilde{A}_1 a_u + \tilde{B}_1 a_u^\dagger + \tilde{C}_1 a_{k_1} + \tilde{D}_1 a_{k_1}^\dagger + \tilde{E}_1 a_{s_1} \tag{35}$$

$$a_{v_2} = \tilde{A}_2 a_u + \tilde{B}_2 a_u^\dagger + \tilde{C}_2 a_{k_2} + \tilde{D}_2 a_{k_2}^\dagger + \tilde{E}_2 a_{s_2} \tag{36}$$

To continue, we note that the k_1 and s_1 modes are not necessarily parallel or orthogonal to the k_2 and s_2 modes. Therefore we have to decompose the modes further to achieve

$$a_{v_1} = A_1 a_u + B_1 a_u^\dagger + C_1 a_{k_1} + D_1 a_{k_1}^\dagger + E_1 a_{s_1} \tag{37}$$

$$a_{v_2} = \cos(\phi)\left(A_2 a_u + B_2 a_u^\dagger + C_2 a_{k_1} + D_2 a_{k_1}^\dagger + E_2 a_{s_1} + F_2 a_{s_1}^\dagger + G_2 a_q + H_2 a_q^\dagger\right) + \sin(\phi) a_p \tag{38}$$

To perform the Bloch-Messiah decomposition, two orthogonal modes must be found, since now the Bloch-Messiah decomposition must be performed over four modes (u, k_1, s_1 and q). The two orthogonal modes and the following Bloch-Messiah decomposition are found numerically. Python code achieving the full transformation of two modes of the output field is provided in ref. [3].

III. SPATIAL MODES AND POLARIZATION MODES

In the main text we discussed temporal/frequency modes but identical results hold accounting for spatial and polarization modes. The Hamiltonian is given by

$$H = \sum_{\mu,\nu} \int\int d\vec{k} d\vec{k}' J(\vec{k},\vec{k}',\mu,\nu) a^\dagger(\vec{k},\mu) a^\dagger(\vec{k}',\nu) + \sum_{\mu,\nu} \int\int d\vec{k} d\vec{k}' L(\vec{k},\vec{k}',\mu,\nu) a^\dagger(\vec{k},\mu) a(\vec{k}',\nu) - \text{h.c}, \tag{39}$$

leading to a Bogoliubov transform of the form

$$a_{\text{out}}(\vec{k},\mu) = \sum_{\nu} \int d\vec{k}' F(\vec{k},\vec{k}',\mu,\nu) a_{\text{in}}(\vec{k}',\nu) + \sum_{\nu} \int d\vec{k}' G^*(\vec{k},\vec{k}',\mu,\nu) a_{\text{in}}^\dagger(\vec{k}',\nu), \tag{40}$$

by utilizing this equation to find $g_1(\vec{k}_1,\mu,\vec{k}_2,\nu)$ we can utilize identical arguments to find that an input single spatiotemporal and polarization mode quantum state will feed into a maximum of two modes on the output.

IV. NONDEGENERATE MODES

Although we show that a single spatiotemporal mode pulse transforming under any second order Hamiltonian will feed only two output modes, in some cases it may be more convenient to decompose the field into more modes. This is the case if for example the signal and idler photons are emitted with very different frequencies or different polarization or direction of propagation. We can in that case redefine the Hamiltonian to

$$H = \int\int d\omega d\omega' J(\omega,\omega')a^\dagger(\omega)b^\dagger(\omega') + \int\int d\omega d\omega' K(\omega,\omega')a^\dagger(\omega)a(\omega') + \int\int d\omega d\omega' L(\omega,\omega')b^\dagger(\omega)b(\omega') - \text{h.c}, \quad (41)$$

leading to a Bogoliubov transform of the form

$$\begin{aligned} a_{\text{out}}(\omega) &= \int d\omega' F_a(\omega,\omega')a_{\text{in}}(\omega') + \int d\omega' G_a^*(\omega,\omega')b_{\text{in}}^\dagger(\omega'), \\ b_{\text{out}}(\omega) &= \int d\omega' F_b(\omega,\omega')b_{\text{in}}(\omega') + \int d\omega' G_b^*(\omega,\omega')a_{\text{in}}^\dagger(\omega'), \end{aligned} \quad (42)$$

where F_a, G_a, F_b, G_b are given by J, K and L. We can thus decompose the g_1 matrix in a similar manner as in the main text, but with 4 modes that do not mix between the a and b modes, and an infinite number of modes that originate from the vacuum field contribution.

$$g_{a,b}^{(1)}(\omega_1,\omega_2) = \underbrace{\sum_{i=1}^{4} n_i v_i^*(\omega_1)v_i(\omega_2)}_{\text{depends on quantum state}} + \underbrace{\sum_{i=1}^{\infty} m_i w_i^*(\omega_1)w_i(\omega_2)}_{\text{independant of quantum state}} . \quad (43)$$

V. DEGENERATE OPO EXAMPLE

Consider the cavity Hamiltonian

$$H = \frac{i\xi(t)}{2}\left(\left(a^\dagger\right)^2 - a^2\right), \quad (44)$$

accompanied by leakage outside the cavity at a rate γ. The Heisenberg equation of motion for the cavity annihilation operator gives:

$$\dot{a} = -\frac{\gamma}{2}a(t) - \sqrt{\gamma}a_{in}(t) + \xi(t)\,a^\dagger(t) \quad (45)$$

where $\sqrt{\gamma}$ is the coupling of the cavity (assumed real and independent of time) and $\xi(t)$ is a time dependent gain (also assumed real). The solution to this differential equation is

$$\begin{aligned} a(t) = e^{-\frac{\gamma}{2}t}\Big\{&\cosh(\chi(t_0,t))a(t_0) + \sinh(\chi(t_0,t))a^\dagger(t_0) \\ &- \sqrt{\gamma}\int_{t_0}^{t} dt' e^{\frac{\gamma}{2}t'}\left[\cosh(\chi(t',t))\,a_{in}(t') + \sinh(\chi(t',t))\,a_{in}^\dagger(t')\right]\Big\} \end{aligned} \quad (46)$$

where t_0 is the initial time, and

$$\chi(t',t) = \int_{t'}^{t} dt_1 \xi(t_1). \quad (47)$$

A. Output modes

For an initial vacuum state of the parametric amplifier, and a vacuum input field, the g_1-function is

$$g_1^{(\text{vac})}(t,t') = \gamma\exp\left(-\frac{\gamma}{2}(t+t')\right)\left(\sinh(\chi(t_0,t))\sinh(\chi(t_0,t')) + \gamma\int_{t_0}^{t_<} dt_1 \exp(\gamma t_1)\sinh(\chi(t_1,t))\sinh(\chi(t_1,t'))\right), \quad (48)$$

where $t_<$ signifies the lesser of t and t'. In a more interesting case, the input field $b_{\text{in}}(t)$ is taken to be in a non-vacuum pulsed input state, with temporal profile $u(t)$. In the following, we consider the case of different quantum states in the input pulse.

1. Input pulse in Fock state

As an initial case, we consider the input pulse to be in a Fock state $|n\rangle$. We wish to calculate the $g_1(t,t')$-function, to find the output temporal modes for this case. Using $\langle a_{in}^\dagger(t)a(t')\rangle = nu^*(t)u(t')$ and $\langle a_{in}(t)a_{in}(t')\rangle = 0$, the result is

$$\begin{aligned} g_1^{(fock)}(t,t') &= nu(t)u(t') - n\gamma e^{-\frac{\gamma}{2}t'}u(t)\int_{t_0}^{t'} dt_1 f(t_1,t') - n\gamma e^{-\frac{\gamma}{2}t}u(t')\int_{t_0}^{t} dt_1 f(t_1,t) \\ &+ n\gamma^2 e^{-\frac{\gamma}{2}(t+t')}\left(\int_{t_0}^{t} dt_1 f(t_1,t)\int_{t_0}^{t'} dt_3 f(t_3,t') + \int_{t_0}^{t} dt_1 g(t_1,t)\int_{t_0}^{t'} dt_3 g(t_3,t')\right) \\ &+ g_1^{(vac)}(t,t'), \end{aligned} \tag{49}$$

where $f(t,t') = u(t)e^{\frac{\gamma}{2}t}\cosh(\chi(t,t'))$ and $g(t,t') = u(t)e^{\frac{\gamma}{2}t}\sinh(\chi(t,t'))$. This can be further rearranged into

$$\begin{aligned} g_1^{(fock)}(t,t') = n&\left(u(t) - \gamma e^{-\frac{\gamma}{2}t}\int_{t_0}^{t} dt_1 f(t_1,t)\right)\left(u(t') - \gamma e^{-\frac{\gamma}{2}t'}\int_{t_0}^{t'} dt_1 f(t_1,t')\right) \\ &+ n\gamma^2 e^{-\frac{\gamma}{2}(t+t')}\int_{t_0}^{t} dt_1 g(t_1,t)\int_{t_0}^{t'} dt_3 g(t_3,t') + g_1^{(vac)}(t,t') \end{aligned} \tag{50}$$

which shows that the contribution from the input pulse will at most split into two modes, as discussed previously. Note however, that the two modes in the above equation are not necessarily orthogonal, and an eigenvalue decomposition is needed to find the actual two orthogonal modes of the output field.

2. Input pulse in a coherent state

Now we consider a coherent state as input $|\alpha\rangle$ in a pulse shape $u(t)$. The analysis is very similar to the case of the Fock state, execpt that the modes are $\langle a_{in}^\dagger(t)a(t')\rangle = |\alpha|^2 u^*(t)u(t')$ and $\langle a_{in}(t)a_{in}(t')\rangle = \alpha^2$. The result is

$$\begin{aligned} g_1^{(coh)}(t,t') = &\left(\alpha^* u(t) - \gamma e^{-\frac{\gamma}{2}t}\left[\int_{t_0}^{t} dt_1(\alpha^* f(t_1,t) + i\alpha g(t_1,t))\right]\right)\left(\alpha u(t') - \gamma e^{-\frac{\gamma}{2}t'}\left[\int_{t_0}^{t'} dt_1(\alpha f(t_1,t') - i\alpha^* g(t_1,t'))\right]\right) \\ &+ g_1^{(vac)}(t,t'). \end{aligned} \tag{51}$$

As can be seen, this leads to only one populated output mode, depending on the input state.

VI. OPA EXAMPLE

To find the mode decomposition for the OPA, we must first find the transformation function $F(\omega,\omega'), G(\omega,\omega')$ from Eq. (3) in the main text. Assuming a parametric down conversion Hamiltonian given by:

$$H = \int_{-L/2}^{L/2} dz\chi^{(2)} E_p^{(+)}(z,t)E_s^{(-)}(z,t)E_s^{(-)}(z,t) \tag{52}$$

from which we can derive a set of integro-differential equations of the form [4]:

$$\begin{aligned} \frac{d}{dz}F(z,\omega_1,\omega_2) &= \int_{-\infty}^{\infty} d\omega' s(z,\omega_1,\omega')G(z,\omega_2,\omega') \\ \frac{d}{dz}G(z,\omega_1,\omega_2) &= \int_{-\infty}^{\infty} d\omega' s^*(z,\omega',\omega_1)F(z,\omega_2,\omega') \end{aligned} \tag{53}$$

with $s(z,\omega_1,\omega_2) = -iD\alpha(\omega_1+\omega_2)\exp(i\Delta k(\omega_1,\omega_2)z)$, $\alpha(\omega)$ is the pump assumed to be in a strong coherent state (non-depleted pump approximation), D collects the constants in front, and $\Delta k(\omega_1,\omega_2) = k(\omega_{\text{pump}}) - k(\omega_1) - k(\omega_2)$.

We solve Eq. (53), with the method presented in ref [5] and code adapted from ref [4]. Then, we input the found functions into our equation for $g^{(1)}(\omega_1,\omega_2)$ and continue to find the mode functions and occupations as described in the main text.

Specifically, for the results in Fig 3b,e, we consider a single photon input in a gaussian shaped mode function for the input quantum seed and a strong gaussian pump. We vary the detuning and their relative width. We consider $D\alpha(\omega) = \frac{40}{\sqrt{2\pi\sigma_p^2}L}\exp\left(-\frac{(\omega-\omega_{\text{pump}})^2}{2\sigma_p^2}\right)$ and $L = 2c/\sigma_u$, and a dispersion relation $k(\omega) = n\omega/c$ where we assume the refractive index is independent of frequency.

VII. TRAVELING WAVE PARAMETRIC AMPLIFIER

In this example we consider a series of OPO amplifiers, each with a constant pump, and a constant decay rate. For each cavity, the Hamiltonian is of the form $H = \Delta a^\dagger a + i\frac{\xi}{2}\left(\left(a^\dagger\right)^2 - a^2\right)$. Following the derivation from [6], the cavity operator can be expressed in frequency space in terms of the input field operators as

$$a(\Delta+\omega) = -\sqrt{\gamma}\frac{(\frac{\gamma}{2}-i\omega)a_{in}(\Delta+\omega)+\xi a_{in}^\dagger(\Delta-\omega)}{(\frac{\gamma}{2}-i\omega)^2-\xi^2} \tag{54}$$

Using the input-output relation $a_{out} = a_{in} + \sqrt{\gamma}a$ we get the output operator as

$$\begin{aligned}
a_{\text{out}}(\Delta+\omega) &= -\frac{\left[\left(\frac{\gamma}{2}\right)^2+\omega^2+\xi^2\right]a_{\text{in}}(\Delta+\omega)+\xi\gamma a_{\text{in}}^\dagger(\Delta-\omega)}{\left(\frac{\gamma}{2}-i\omega\right)^2-\xi^2}\\
a_{\text{out}}^\dagger(\Delta-\omega) &= -\frac{\left[\left(\frac{\gamma}{2}\right)^2+\omega^2+\xi^2\right]a_{in}^\dagger(\Delta-\omega)+\xi\gamma a_{in}(\Delta+\omega)}{\left(\frac{\gamma}{2}-i\omega\right)^2-\xi^2}
\end{aligned} \tag{55}$$

such that, going to the quadrature operators $X(\Delta+\omega) = \frac{a(\Delta+\omega)+a^\dagger(\Delta-\omega)}{2}$ and $P(\Delta+\omega) = \frac{a(\Delta+\omega)-a^\dagger(\Delta-\omega)}{2i}$, we get

$$\begin{aligned}
X_{\text{out}}(\Delta+\omega) &= -\frac{\left[(\frac{\gamma}{2}+\xi)^2+\omega^2\right]}{\left(\frac{\gamma}{2}-i\omega\right)^2-\xi^2}X_{in}(\Delta+\omega) \equiv \chi_X X_{in}(\Delta+\omega),\\
P_{\text{out}}(\Delta+\omega) &= -\frac{\left[(\frac{\gamma}{2}-\xi)^2+\omega^2\right]}{\left(\frac{\gamma}{2}-i\omega\right)^2-\xi^2}P_{\text{in}}(\Delta+\omega) \equiv \chi_P P_{in}(\Delta+\omega)
\end{aligned} \tag{56}$$

From which we can deduce that:

$$\begin{aligned}
X_{\text{N}}(\omega) &= (\chi_X)^N X_0,\\
P_{\text{N}}(\omega) &= (\chi_P)^N P_0,
\end{aligned} \tag{57}$$

where the N denotes the output after N OPO amplifiers in series, and X_0 and P_0 are the quadrature operators of the input field to the series of OPO amplifiers. The output field operators are found from inverting the definition of the quadrature operators

$$\begin{aligned}
&a_{out,N}(\Delta+\omega) = \frac{\chi_X^N+\chi_P^N}{2}a_{in}(\Delta+\omega)+\frac{\chi_X^N-\chi_P^N}{2}a_{in}^\dagger(\Delta-\omega) = \Gamma_N^+ a_{in}(\Delta+\omega)+\Gamma_N^- a_{in}^\dagger(\Delta-\omega)\\
&a_{out,N}^\dagger(\Delta-\omega) = \Gamma_N^+ a_{in}^\dagger(\Delta-\omega)+\Gamma_N^- a_{in}(\Delta+\omega)
\end{aligned} \tag{58}$$

With this result we get the Bogoliubov transformation functions

$$F(\omega,\omega') = \Gamma_N^+(\omega-\Delta)\delta(\omega'-\omega) \tag{59}$$

$$G^*(\omega,\omega') = \Gamma_N^-(\omega-\Delta)\delta(\omega'-2\Delta+\omega) \tag{60}$$

which can be inserted in the main paper to recover the Bogoliubov transformation and to find the g_1-function.

VIII. ADDITIONAL LOSSES

Losses may occur in transmission or inside the amplifying medium, and can be modelled accordingly. To illustrate the main effects of losses, we consider here a reduction factor $\sqrt{1-l^2}$ on the output amplitude and an accompanying uncorrelated (vacuum) Langevin noise term $\sqrt{l}a_p$ in Eq. (8) in the main text,

$$\begin{aligned} a_{v,\text{out}} = \sqrt{1-l^2}(&\zeta \langle f,u\rangle a_u + \xi \langle u,g\rangle a_u^\dagger \\ &+ \zeta\sqrt{1-|\langle f,u\rangle|^2}\langle h,k\rangle a_k + \xi\sqrt{1-|\langle u,g\rangle|^2}a_k^\dagger \\ &+ \zeta\sqrt{1-|\langle f,u\rangle|^2}\sqrt{1-|\langle k,h\rangle|^2}a_s) + \sqrt{l}a_p. \end{aligned} \tag{61}$$

We plot in Figure 1 below the resulting degraded fidelity of a squeezed cat state with parameters like in the leftmost panel of Figure 4c of the main text.

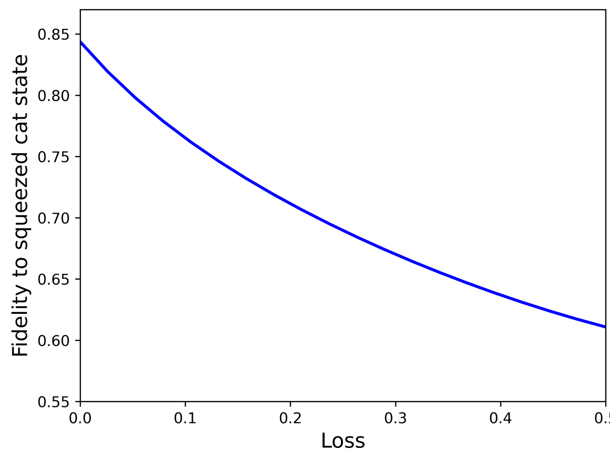

Figure 1. **Fidelity vs loss of a squeezed Schrodinger cat state**.

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