Parametric amplification of a quantum pulse

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Creating and manipulating quantum states of light requires nonlinear interactions. We present here a multimode theory for the transformation of an arbitrary quantum pulse by Hamiltonians that are quadratic in field creation and annihilation operators. We show, in particular, that any input quantum pulse will feed only one or two distinct output modes. Our result readily provides both the output modes and the quantum states after transformation of arbitrary input wave packets. While being central for applications in quantum information processing with traveling bosonic modes, e.g., in quantum networks, our theoretical method and its results are not anticipated by the conventional Bogoliubov diagonalization of the problem in many input and output eigenmodes.

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I. INTRODUCTION

Quantum states of light are the key ingredients for some of the most promising quantum technologies [1]. Light pulses can propagate between stationary components in quantum networks [2], and their applications include quantum key distribution [3], sensing beyond the standard quantum limit [4–6], bosonic-error-correcting codes [7,8], and measurement-based quantum computation [9]. However, the preparation and manipulation of quantum states of light are more complicated for traveling fields than for their stationary counterparts, such as the field in a single-mode cavity. This is because propagation of light explores a continuum of frequency modes [10].

The multimode nature of quantum light is of critical importance in any system involving nonlinearities, as in this case the transformation of a quantum state will generally lead to a population of output pulse shapes correlated with the photon number state content [9,10]. The output field is thus inherently multimode in nature, and the quantum properties of the state may not be accessible to the desired quantum information processing task.

In this article, we deal with operations on traveling light pulses, governed by Hamiltonians that are of second order in the creation and annihilation operators. This class of Hamiltonians can describe beam splitters and interferometers, dispersion, diffraction, and polarization rotation, but also parametric amplification, parametric down-conversion, and frequency conversion.

We are particularly interested in parametric amplification which is in the single (cavity)-mode case described by the Hamiltonian ($\hbar = 1$) $H = \frac{i\xi}{2}[(a^\dagger)^2 - a^2]$ and the time evolution in the Heisenberg picture:

$$a(t) = \cosh(\xi t)a(0) + \sinh(\xi t)a^\dagger(0). \tag{1}$$

This transformation is also referred to as squeezing, as it reduces the value (and uncertainty) of one of the field quadratures while the other quadrature is correspondingly amplified [11].

Since their initial demonstration [12–14], squeezing and parametric amplification have been ubiquitous tools in quantum optics. The creation of squeezed states with high photon counts per mode have led to the adoption of pulsed pumps for vacuum squeezing [14–16], which in turn has inspired methods for multimode analysis of a squeezed vacuum [17–20]. Squeezed states are employed for quantum sensing [6] and a multimode squeezed vacuum is utilized to generate resource cluster states for measurement-based quantum computation [21,22] and Gaussian boson sampling [23,24].

However, it is also desirable to parametrically amplify nonvacuum and particularly non-Gaussian quantum states. In its single-mode version, parametric amplification has been proposed for quantum state tomography [25,26], for signal amplification and readout of superconducting qubits [27,28] and for ultrafast nonlinear nanophotonics [29]. Furthermore, it is part of the gate set for continuous-variable quantum computing [30] and for creation and manipulation of Gottesman-Kitaev-Preskill (GKP) states for quantum error correction [31,32]. As there are now proposals and attempts to employ these schemes with traveling wave-packet modes [33,34], it is pertinent to ask to what extent Eq. (1) applies to the transformation of the quantum state contents of a pulse of radiation.

Previous works have identified a set of input eigenmodes that all undergo single-mode squeezing [35–37]. While this, indeed, offers an expression for the parametric amplification

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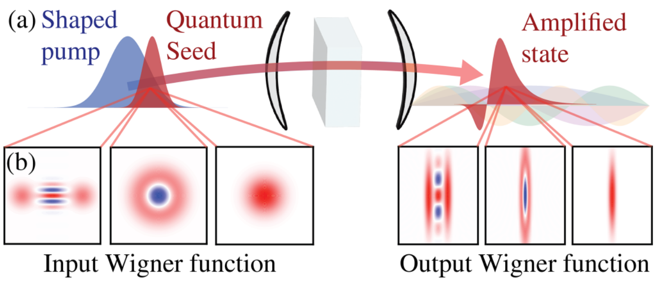

FIG. 1. Parametric amplification of quantum pulses. (a) A nonlinear medium in a cavity is driven by an arbitrarily shaped classical pump (blue) such that a quantum pulse (red), transmitted through the cavity, is subject to parametric amplification. The amplified quantum seed populates only two spatiotemporal output modes that are further mixed with squeezed-vacuum and vacuum contributions. (b) When the input pulse is prepared in a Schrödinger cat state, a single-photon state, and a vacuum state (shown in the left panels), the most occupied output mode is in an approximate, amplified (squeezed) version of these input states (shown in the right panels).

of the continuous input field, its practical application remains very complex when applied to quantum states occupying arbitrary wave packets. As any such wave packet must be expanded on the input eigenmodes, the outcome may be highly entangled across the corresponding output modes, and characterizing the quantum state has a complexity similar to that of the famous boson sampling problem [38] for which there is no known algorithm in polynomial time. Thus, an efficient method for calculating the transformation of an arbitrary input pulse through a Hamiltonian of second order in the creation and annihilation operators remains elusive.

Here we present a direct calculation of the transformation of quantum pulses of radiation subject to optical components that actuate quadratic Hamiltonians, such as dispersive and parametrically amplifying elements. Surprisingly, we find that any input pulse feeds at most two output modes and that there exists a family of quantum states, including some Schrödinger cat and coherent states, that feed only one output mode. For these states, irrespective of the shape of the pulse, we recover a squeezed version of the initial state mixed with squeezed-vacuum components.

Figure 1 shows examples of parametric amplification of incident wave-packet states. The lower panels in Fig. 1 show Wigner functions of the input states and the states of the most populated output mode. Our theory shows that the parametric amplifier produces an approximate single-mode squeezing transformation for a wide family of input quantum states, even if the input pulse is a complicated superposition of the eigenmodes of the amplifier.

II. QUANTUM STATE TRANSFORMATION BY A PARAMETRIC AMPLIFIER

In this section we determine the output of the amplifier given any arbitrary quantum state occupying an arbitrary input wave packet with the frequency representation $u(\omega)$. We treat here the case of guided propagation of a single transversal field mode, while further spatial and polarization degrees of freedom are readily included (see Supplemental Material Sec. SIII in [39]). We define $a^\dagger_{u,\text{in}} = \int_{-\infty}^{\infty} u(\omega)a^\dagger(\omega)d\omega$, where $a(\omega)$ are bosonic operators with the commutation relation $[a(\omega), a^\dagger(\omega')] = \delta(\omega-\omega')$.

We consider a general quadratic Hamiltonian in the creation and annihilation operators [37]:

$$H = \iint d\omega d\omega' K(\omega,\omega')a^\dagger(\omega)a(\omega') + \iint d\omega d\omega' J(\omega,\omega')a^\dagger(\omega)a^\dagger(\omega') + \text{H.c.}, \tag{2}$$

where $K(\omega,\omega')$ alone leads to a transformation of the input wave packet, while $J(\omega,\omega')$ is the amplitude for creating a pair of photons in the output field at frequencies ω and ω', and H.c. denotes the Hermitian conjugate. While we consider only Hamiltonians of second order, master equation methods have been recently developed to describe interactions of any nonlinear type between quantum pulses and finite scatterers [40–42]. These works rely on numerical solution of cascaded master equations to identify both the relevant output modes and states. We envision that the analytical results in the present work and the numerical approaches may be combined in systems with cascaded quadratic and higher-order nonlinearities.

The Heisenberg equations of motion for the field creation and annihilation operators can be solved to give the following transformation [43]:

$$a_{\text{out}}(\omega) = \int d\omega' F(\omega,\omega')a_{\text{in}}(\omega') + \int d\omega' G^*(\omega,\omega')a^\dagger_{\text{in}}(\omega'), \tag{3}$$

where $F(\omega,\omega')$ and $G(\omega,\omega')$ are uniquely determined by the functions $J(\omega,\omega')$ and $K(\omega,\omega')$ [37]. We give examples in later sections of the article.

The ideal transformation for parametric amplification of a single-mode quantum pulse would amount to a single output mode squeezed as in Eq. (1). However, since the amplification applies to a continuum of field modes and is frequency dependent, the output field may occupy mode functions that will generally differ from the input mode. We therefore move our focus to the calculation of the quantum state content of any given output mode with the creation operator $a^\dagger_{v,\text{out}} = \int_{-\infty}^{\infty} v(\omega')a^\dagger_{\text{out}}(\omega')d\omega'$. First, we analyze the mode content of the output to find the optimal candidate v function, i.e., the one that best captures the transformed input quantum state. In the subsequent sections, we calculate its actual quantum state content.

A. Modes at the output of the amplifier

To characterize the output field, we evaluate the first-order coherence function $g_1(\omega_1,\omega_2) = \langle a^\dagger_{\text{out}}(\omega_1)a_{\text{out}}(\omega_2)\rangle$ [10]. Consider the situation where the input occupies a single wave packet, $u(\omega)$, with a quantum state given by a density matrix, ρ_u, and vacuum in all modes orthogonal to $u(\omega)$. We then have $\langle a^\dagger_{\text{in}}(\omega')a_{\text{in}}(\omega'')\rangle = \langle a^\dagger_u a_u\rangle\, u^*(\omega')u(\omega'')$ and similar expressions for the other input field correlation functions.

With Eq. (3), we find

$$g_1(\omega_1,\omega_2)=\langle a_u^\dagger a_u\rangle\int d\omega' F^*(\omega_1,\omega')u^*(\omega')\int d\omega'' F(\omega_2,\omega'')u(\omega'')+\langle a_u^\dagger a_u^\dagger\rangle\int d\omega' F^*(\omega_1,\omega')u^*(\omega')\int d\omega'' G^*(\omega_2,\omega'')u^*(\omega'')$$
$$+\langle a_u a_u\rangle\int d\omega' G(\omega_1,\omega')u(\omega')\int F(\omega_2,\omega'')u(\omega'')d\omega''+\langle a_u^\dagger a_u\rangle\int d\omega' G(\omega_1,\omega')u(\omega')\int d\omega'' G^*(\omega_2,\omega'')u^*(\omega'')$$
$$+\int d\omega' G(\omega_1,\omega')G^*(\omega_2,\omega'). \tag{4}$$

We observe that the coherence function is governed by both the input pulse's shape and quantum state. The first four terms contain correlation functions of the input field (for example, $\langle a_u a_u\rangle$) while the last term is the squeezed-vacuum output of the amplifier in the absence of any input pulse.

By inspection of Eq. (4), it turns out that the input quantum pulse gives rise to population of only two output modes, while a potentially infinite number of modes are populated by a squeezed vacuum,

$$g_1(\omega_1,\omega_2)=\underbrace{\sum_{i=1}^{2}n_i v_i^*(\omega_1)v_i(\omega_2)}_{\text{dependent on input quantum state}}+\underbrace{\sum_{i=1}^{\infty}m_i w_i^*(\omega_1)w_i(\omega_2)}_{\text{independent of input quantum state}}. \tag{5}$$

This is demonstrated in detail in Sec. SI of Ref. [39].

In the special case of a coherent state input ($\rho_u=|\alpha\rangle\langle\alpha|$), the operator expectation values factor, and the above expansion leads to only a single output mode seeded by the input field. For a Fock state input ($\rho_u=|n\rangle\langle n|$), the output field occupies two modes, seeded by the input pulse. In fact, the condition for which the output occupies only a single mode is [39]

$$\langle a_u^\dagger a_u\rangle=|\langle a_u a_u\rangle|. \tag{6}$$

This condition is a main result of our analysis. An input field fulfilling this condition will have only one mode of the output field that depends on the input field. This does not mean, however, that we obtain single-mode squeezing like Eq. (1) or that this result reduces to an eigenmode squeezing situation like in Refs. [35–37]. As we see in the following section, the quantum state will in general be polluted by squeezed-vacuum and vacuum terms.

B. The quantum state in an output mode

While the operation of the amplifier is fully described by the multimode Bogoliubov transformation (3), this does not provide an immediate description of the transformation of a given quantum state input. We are faced, in fact, with an instance of the boson sampling problem [38], and the number state representation of the output state is an unwieldy expression in terms of matrix permanents and the density matrix of the input state. The problem simplifies, however, when we restrict our interest to the quantum state of the output of the amplifier in any single mode $v(\omega)$. Our theory applies to any output mode of interest, but a good choice for $v(\omega)$ is the most occupied single-mode function at the output, say, $v_1(\omega)$ in Eq. (5). By applying Eq. (3), we can formally write

$$a_{v,\text{out}}=\int v^*(\omega)a_{\text{out}}(\omega)d\omega=\zeta a_{f,\text{in}}+\xi a_{g,\text{in}}^\dagger, \tag{7}$$

where we have defined two new input-mode functions through $f^*(\omega)=\int v^*(\omega')F(\omega',\omega)d\omega'/\zeta$ and $g(\omega)=\int v^*(\omega')G^*(\omega',\omega)d\omega'/\xi$, and ζ and ξ ensure their normalization (we omit the subscript "in" in the following).

Notice a few key properties of this transformation. First, in the purely dispersive case, where $\xi=0$ and $\zeta=1$, the annihilation operator of the output-mode function $v(\omega)$ represents the exact same quantum state content as the one that occupied the input-mode function $f(\omega)$ [which is not necessarily $u(\omega)$]. This reflects that quantum states are unchanged while traveling through a linearly dispersive element, but their mode functions change shape.

Second, in the amplifier case, where $\xi\neq 0$, the functions f and g are normalized but generally not orthogonal, and their relationship with the populated input mode $u(\omega)$ is not yet specified. To find the output quantum state in the mode $v(\omega)$, we must decompose the transformation into one that refers specifically to the input mode $u(\omega)$ and vacuum modes orthogonal to $u(\omega)$. Defining $\langle f,g\rangle=\int f^*(\omega)g(\omega)d\omega$, we decompose the modes into parallel and orthogonal components of the input mode u,

$$\begin{aligned}a_{v,\text{out}}=&\,\zeta\langle f,u\rangle a_u+\xi\langle u,g\rangle a_u^\dagger\\&+\zeta\sqrt{1-|\langle f,u\rangle|^2}\langle h,k\rangle a_k+\xi\sqrt{1-|\langle u,g\rangle|^2}a_k^\dagger\\&+\zeta\sqrt{1-|\langle f,u\rangle|^2}\sqrt{1-|\langle k,h\rangle|^2}a_s,\end{aligned} \tag{8}$$

with mode functions given by (for more information see Sec. SII in Ref. [39])

$$k(\omega)=\frac{g(\omega)-u(\omega)\langle u,g\rangle}{\sqrt{1-|\langle u,g\rangle|^2}},$$
$$h(\omega)=\frac{f(\omega)-u(\omega)\langle u,f\rangle}{\sqrt{1-|\langle u,f\rangle|^2}},$$
$$s(\omega)=\frac{h(\omega)-k(\omega)\langle k,h\rangle}{\sqrt{1-|\langle k,h\rangle|^2}}. \tag{9}$$

Equation (8) is a main result of our analysis. It shows how a single output mode captures a squeezed version of the potentially interesting state occupying the input pulse (first line). However, it is mixed with a squeezed-vacuum component (second line) and an unsqueezed-vacuum component (third line). In the absence of the terms in the second and third lines of Eq. (8), we recover unitary single-mode squeezing as in Eq. (1), which corresponds to the input being in one of the eigenmodes of the system, as described by previous works on multimode squeezing [35–37]. Here we have found the transformation for an arbitrary input mode, which includes the terms in the second and third line in Eq. (8), which will in general contribute added quantum noise. The vacuum noise component a_s ensures unitarity in connection with the loss into other temporal and spatial modes, while additional loss in the transmission or amplification processes can be readily included by amplitude reduction factors and further vacuum noise terms (see Sec. SVIII in Ref. [39]).

It is interesting to compare Eq. (8) with the results of the mode decomposition of the parametric amplification [35]. Instead of decomposing the Bogoliubov transformation of the multimode field operators into single-mode squeezing transformations that act separately on a complete set of orthogonal input eigenmodes as in Refs. [35,44], we have contrarily shown that a state occupying any single wave-packet mode will always be amplified into only a single output mode [if its quantum state obeys Eq. (6)] or two modes with an additional squeezed-vacuum noise contribution. The method presented and our calculation of output quantum states apply for any chosen input and output modes. Most importantly, our theory provides the transformation of arbitrary quantum states, available for subsequent quantum information tasks where previous theory has aimed mainly to characterize the gain and quantum noise in amplified signals.

The transformation in Eq. (8) is effectively described by the Bloch-Messiah reduction [44,45], which states that a general multimode Bogoliubov transformation can be separated into three steps: first, a linear beam-splitter transformation among the modes, then a sequence of single-mode squeezing operations, and finally another beam-splitter transformation. We thus find $\rho_{v,\text{out}} = \text{Tr}_{k,s}(U\rho_u U^\dagger)$, with

$$U = U_{u,s}(\theta_3,\phi_3)U_{u,k}(\theta_2,\phi_2)S_u(r_1)S_k(r_2)U_{u,k}(\theta_1,\phi_1), \quad (10)$$

where the beam-splitter (U) and squeezing (S) transformations and their parameters are specified in Sec. SII in Ref. [39]. Note that these transformations act in a particularly simple manner in the Wigner function representation of quantum states, where they amount to linear transformations on the field quadrature variables [46], given directly by the coefficients in Eq. (8). By a straightforward extension of this procedure, one can also find the joint quantum state of the pair of output modes v_1 and v_2 fed by the input mode and, e.g., study their mutual entanglement properties (see Sec. SII in Ref. [39] and the code repository) [47]. In the following, we restrict the analysis to the most occupied single mode v_1 for v_{out}, and we leave the study of other candidate output modes and the two-mode output for later investigation.

III. EXAMPLES

In this section we consider applications of our theory. We first provide two examples for the use of pulsed parametric amplification as a tool for quantum state transformation, and then we provide more detailed calculations for specific systems.

The first example involves continuous-variable quantum computation and error correction with GKP states [7]. Recently experiments have generated small GKP states in the optical range [8]. Generating and error correcting with GKP states often requires squeezing of the states with minimal infidelity [48–50]. With previous methods, analyzing how an arbitrary input pulse would transform under this parametric amplification or squeezing process was not possible. With our new method of calculation, these experiments can be analyzed and the operations can be optimized.

The second example involves parametric amplification for tomography of quantum states. Tomography of quantum states is challenging because any loss will impair the fidelity of the state. Here, parametric amplification can be used to amplify the quantum state, and then the measurement can be done in a manner less sensitive to losses [25,29]. Previous experiments have shown this process only for Gaussian states, where the effect of loss can be taken into account by other means. With our method, we have shown that the amplified output populates only two relevant modes, and by measuring the state of these modes, our theory may be used to reconstruct any input quantum state.

A. Amplification by an optical parametric oscillator

To demonstrate applications of our theory, we consider as a first example a degenerate optical parametric oscillator (OPO) cavity with the cavity Hamiltonian

$$H_{\text{sys}} = \Delta a^\dagger a + \frac{i\xi(t)}{2}[a^{\dagger 2} - a^2], \quad (11)$$

where Δ is the detuning of the cavity and $\xi(t)$ is the time-dependent parametric gain due to a pulsed drive. We assume an input quantum state occupying a single mode of temporal shape $u(t)$, coupled with the coherent amplitude $\sqrt{\gamma}$ to the cavity mode (the cavity loss rate is γ). Under the assumption of a Markovian coupling, we can relate the output field to the cavity field at time t via the input-output relation [51] $a_{\text{out}}(t) = a_{\text{in}}(t) + \sqrt{\gamma}a(t)$ [52], and solving the Heisenberg equation of motion for the intracavity field, we obtain the explicit Bogoliubov transformation of the field operators in the time domain (see Sec. SV of Ref. [39]). This yields directly the time domain correlation function $g_1(t_1,t_2)$, and the minimal basis of temporal modes, that efficiently describes the parametrically amplified input quantum state.

We analyze first the output of the OPO with only the vacuum seed and a time-dependent Gaussian pump $\xi(t) = \frac{\xi_0}{\sqrt{2\pi\sigma_\xi^2}}\exp\left(-\frac{(t-t_0)^2}{2\sigma_\xi^2}\right)$. Figure 2(a) shows the occupation of the modes of the output field. For a short pulsed pump ($\gamma\sigma_\xi \ll 1$), only a single mode of the output field is excited (squeezed), as can be seen by the large occupation of the most occupied mode in the left-hand side of Fig. 2(a). The mode occupied represents the exponential decay (shown in the inset) of the abruptly excited cavity mode. For wider pumps, the output field populates many modes [shown in the right-hand inset of Fig. 2(a)]. In the limiting case of an infinite duration constant pump, the photons possess strong frequency correlations, $\omega_1 + \omega_2 = \omega_{\text{pump}}$, to conserve energy. In this limit, the

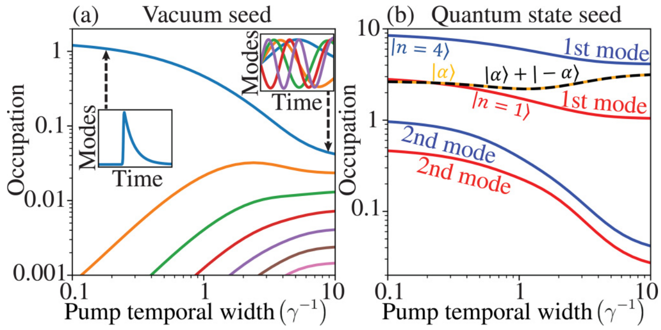

FIG. 2. Temporal modes emitted by a parametric amplifier. (a) Vacuum input. Occupation (expectation value of the photon number operator in a given mode $\langle a_u^\dagger a_u \rangle$) of the most populated modes as a function of pump duration for Gaussian pumps with a constant pulse area $[\int \xi(t)dt, \xi_0 = 1]$. A short pump pulse excites the cavity mode into a squeezed state that leaks into a single-mode traveling pulse. A longer pump pulse overlaps the emission process, and more modes become occupied. The mode shapes for the limiting cases of very wide and narrow pumps are plotted as insets. (b) Occupation of the modes fed by a Gaussian pump ($\xi_0 = 1.5$) and different initial quantum states in the input pulse. The occupation of the modes seeded by an input pulse of duration $\tau = 1/\gamma$ (two modes for Fock states, one mode for coherent and Schrödinger cat ($\alpha = 2$) states) are plotted as a function of the pump width, assuming a constant pump-pulse area. We assume that the pump and quantum pulses peak at the same time.

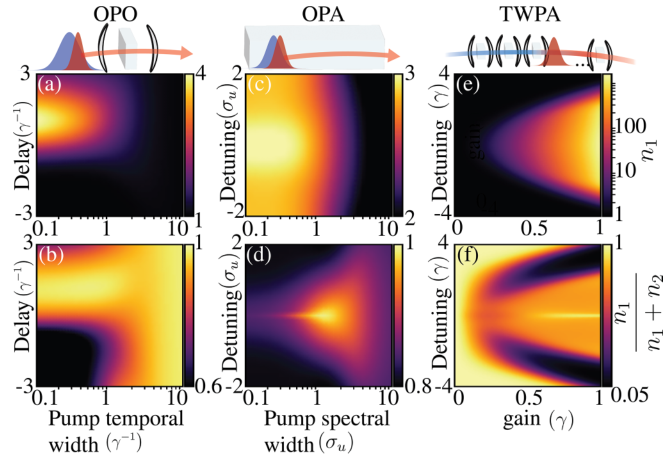

FIG. 3. Comparison of three parametric amplifiers acting on a single-photon input state. Upper panels: Mean photon number in the dominant output mode n_1. Lower panels: Occupation ratio $n_1/(n_1 + n_2)$ as defined in Eq. (4). (a), (b) For the OPO, these quantities are plotted as a function of the delay time between the pump and seed and the pump temporal width in units of the cavity decay rate γ. (c), (d) For the OPA, with a spatially extended nonlinear medium and no cavity, they are shown as a function of the pump-seed detuning and the pump spectral width in units of the spectral width of the input quantum pulse σ_u. (e), (f) For the TWPA, the pump is a continuous wave and the occupations are shown as a function of the pump detuning in units of the cavity decay rate γ and the total amount of gain for 3000 OPOs. For all panels of this plot, the seed is a single photon state in a Gaussian temporal wave packet.

output field explores a continuum of modes and $g^{(1)}(\omega_1, \omega_2) \propto S(\omega_1)\delta(\omega_{\text{pump}} - \omega_1 - \omega_2)$, where $S(\omega_1)$ is the output spectrum [53].

Next, we consider the parametric amplification of a quantum input pulse in the mode $u(t) \propto \exp\left(-\frac{(t-t_0)^2}{2\tau^2}\right)$ in Fig. 2(b). Notice that the coherent state and the Schrödinger cat input feed only into one output mode as they obey the condition in Eq. (6) [54]. In contrast, Fock states lead to a population of two modes, as shown by the red and blue pairs of curves in Fig. 2(b). The efficiency of the gain is largest for the case of a short pump. In this case, the emission by the cavity is negligible during the gain process, and the accumulated cavity field stimulates the strongest gain.

B. Comparison of three parametric amplifiers

We recall that the output of the OPO contains both the transformed input quantum pulse(s) and components of the multimode squeezed vacuum. This results from our general analysis and applies to any setup with quadratic Hamiltonians. We consider exemplary systems and analyze the output field modes that are seeded by a single-photon input pulse, $u(t) \propto \exp\left(-\frac{t^2}{2\sigma_u^2}\right)$. The three setups are illustrated above the upper panels in Fig. 3: (i) the OPO as explained above in Eq. (11), (ii) the optical parametric amplifier (OPA), with a spatially extended nonlinear medium and no cavity, and (iii) the traveling wave parametric amplifier (TWPA), represented here as a concatenated sequence of OPOs. For all three systems, we find regimes of close to ideal single-mode amplification, as shown by the regimes of large occupation in the dominant mode v_1 (upper panels in Fig. 3) and the large ratio between the occupation in the dominant mode n_1 and the sum of the two most populated modes $n_1 + n_2$ (lower panels in Fig. 3).

For the amplification of a single-photon Gaussian pulse by the OPO, a short delay between the short pump and quantum state seed is optimal, for the OPA, a large overlap in the frequency domain between the pump and the seed will provide high gain and single-mode operation, while for the TWPA, we find that resonant cavities provide the highest gain and highest single-mode operation ratio. More information on the calculations presented in Fig. 3 is given in Secs. SV–SVII of Ref. [39].

Next, we implement the theory to calculate the quantum state content of the most occupied mode. Results of these calculations are shown in Fig. 4 for squeezing by an OPO. The output Wigner functions in the most occupied single mode are plotted for different input states: vacuum (a), a single-photon state (b), and a cat state (c) (squeezed cat states can be mixed to create GKP states [7,31]). Results are shown for different pump widths, assuming the optimal delay identified in Fig. 3. We observe the effect of parametric amplification for short pulses while, in agreement with Fig. 2, for longer pump pulses, the quantum states experience little or no amplification. When the pump area is increased by a factor of 5 (rightmost column of Fig. 4), the gain is stronger also for the longer pulses. However, in this regime the quantum states are polluted by the multimode squeezed vacuum generated by the OPO.

To make a quantitative comparison of the transformation of quantum pulses with the ideal single-mode squeezing

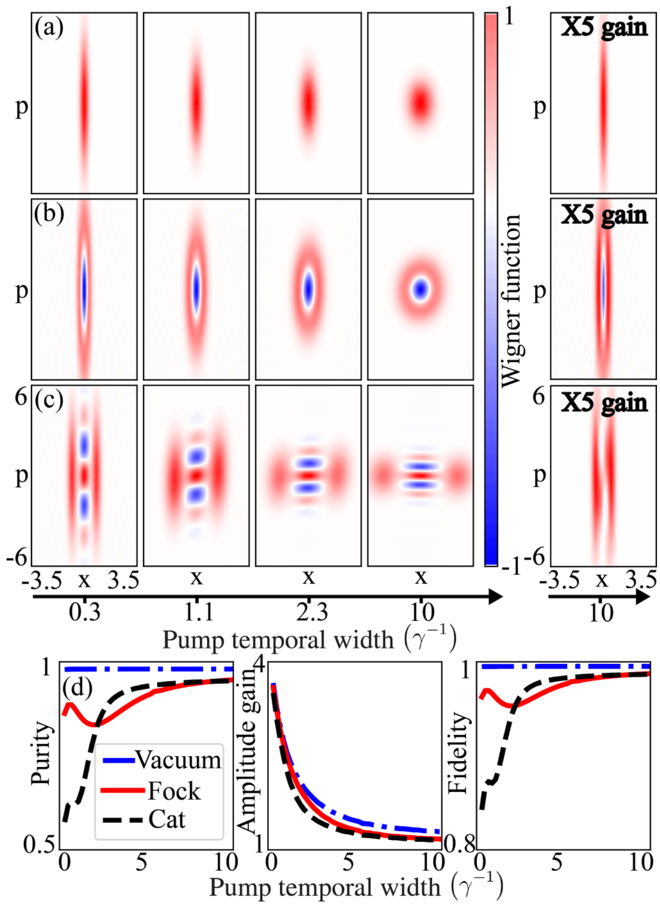

FIG. 4. Transformation of quantum states by an OPO. Wigner functions are shown for the most occupied output temporal modes for different input states: (a) the vacuum state $|0\rangle$, (b) the single-photon Fock state $|n = 1\rangle$, and (c) the Schrödinger cat state $|\alpha\rangle + |-\alpha\rangle$ with $\alpha = 2.5$. Assuming the same pump-pulse area ($\xi_0 = 1.5$), short pulses lead to the strongest squeezing. Yet, when we use long pulses with stronger gain ($\xi_0 = 7.5$), as plotted in the rightmost panels, the states decohere due to the mode entanglement and thermal components, as shown by the smaller amount of Wigner negativity in these panels. (d) The purity of the state in the mode v_1 is plotted on the left. For each pump temporal width, we calculate the quantum state in the mode v_1, then we find the single-mode parametrically amplified state to which it has the highest fidelity, and we plot this fidelity and the corresponding squeezing in panel (d) (middle and right panels). The squeezing is quantified by the amplitude gain in the p quadrature.

transformation, we calculate the quantum state occupying the mode v_1, and in the leftmost panel of Fig. 4(d), we plot the purity of this state, which can be lower than unity, both due to entanglement with the second output mode seeded by the input state v_2 and due to contamination by the squeezed-vacuum modes w_i. In the rightmost panel of Fig. 4(d), we compare the state in v_1 to the input state squeezed by the single-mode transformation of Eq. (1), optimizing over the amount of squeezing, parametrized as gain in the p quadrature, which we plot in the central panel of Fig. 4(d). Notably, when computing the fidelity $\mathcal{F}(A, B) = \mathrm{Trace}(\sqrt{\sqrt{A}B\sqrt{A}})$ [55,56], we find pure state fidelities of over 85% with more than a factor 3 of parametric amplification on the p quadrature, while close to unity fidelity parametric amplification of a single-mode pulse is only possible for much smaller gains, except for the special case of vacuum input. We note that we considered an input seed and pump in a Gaussian temporal mode and that optimizing over these may increase the fidelity, purity, and squeezing.

IV. CONCLUSION

An optical component described by a multimode field Hamiltonian that is quadratic in creation and annihilation operators causes linear dispersion of wave packets combined with correlated creation and annihilation of photon pairs over all modes. A multimode Bogoliubov transformation accounts for the effect of the Hamiltonian on the input field operators. We have shown that, if all but a single mode of the input field are in the vacuum state, despite the multimode character of the problem, only two output modes will contain quantum states that depend on the nontrivial input state. These output modes will generally also contain components of a squeezed vacuum from the parametric amplifier. While these results are at variance with the ideal parametric amplification as a unitary operation on a single-mode quantum field, our analysis can identify optimal parameter settings for reducing additional noise and spreading over more than a single mode.

Our theory applies also to multiple input and output modes, and the joint quantum state of two or more modes can be readily found. Optimization of the fidelity of the parametric amplification process and the more general properties of the output quantum state of the multimode Bogoliubov transformation constitute a promising topic for further study.

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