Supplementary Material: Experimental Observation of Acceleration-Induced Thermality

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I. SUPPLEMENTARY MATERIAL

A. Experimental Methods

To generate our figures, the power spectrum was scaled by $s\frac{3x_0}{4c}$ with $x_0 = 9.37$ cm for positrons in silicon. This scaling factor is included in our analysis since it is used by the experimental group [32] to take into account the detectors signal conversion process to ensure a single photon spectrum. We best fit the acceleration with a factor $\tilde{a}$ and our energy gap is given by a general polynomial of the form $\Delta E = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3$. We then performed a least squares best fit, with cutoffs at multiples of 10 GeV, to obtain the values for our six parameters s, $\tilde{a}$, and a_i. In the case of no energy gap we merely set $\Delta E = 0$. To directly compare to the crystal data set [32], our spectrum was multiplied by a factor of $\frac{3x_0}{4c} = 2.34 \times 10^{-10}$ s. This means the power spectrum we plot is $\frac{d\mathcal{S}}{d\omega} \to s\frac{3x_0}{4c}\frac{d\mathcal{S}}{d\omega}$. Note, here we explicitly put in the speed of light. We then plot our power spectrum with the best fit parameters for the first cutoff to satisfy the chi-squared criterion. To compute the chi-squared statistic, we evaluated our best fit power spectra at the x-value of each data point to compare the theoretical y-value to the data. The chi-squared per degree of freedom is then given by $\chi^2_{the}/\nu = \sum_i \frac{(y_{the}(x^i_{exp}) - y^i_{exp})^2}{\sigma_i^2}$. Here i labels the data points and σ_i is the experimental error of each data point. The number of degrees of freedom is given by $\nu = n - p$, with $n = 150$, $p_{\Delta E} = 6$, and $p_0 = 2$. Here, $p_{\Delta E}$ and p_0 are the number of fit parameters for the power spectrum with and without the energy gap respectively.

In order to analyze the area-entropy law with a low energy cutoff in the energy radiated let us define $\Delta\tilde{E}(\omega_c, \omega_f) = \frac{4c}{3x_0}\int_{\omega_c}^{\omega_f} \frac{d\mathcal{S}_{data}}{d\omega} d\omega dt$. Here, ω_c and ω_f are the cutoff frequency and the final frequency of the emitted photons respectively. Then, we will have the energy radiated given by $\Delta\tilde{E}(\omega_c, 150)$, the initial energy given is by $E_i = 178.2\ GeV - \Delta\tilde{E}(0, \omega_c)$, and the final energy is given by $E_f = 178.2\ GeV - \Delta\tilde{E}(0, 150)$. This parameterization ensures that all energy radiated below the cutoff frequency does not get included in the analysis. Finally, we assume the emission is time independent when evaluating the time integral of the power spectrum to be used in the energy emission $\Delta\tilde{E}(\omega_c, \omega_f)$, i.e. to integrate over time, we simply multiply the power spectrum by the total crystal crossing time. This is based on the time independence of the acceleration. However, we must take into account the thermalization time of the highest frequency emitted. From Fig. 1(a), the thermalization time of highest frequency photon emitted, 150 GeV, is approximately $\tau_{150} = 8.45 \times 10^{-12}\ s$. Then, considering the 3.8 mm thickness of the crystal, the total integration time is given by $\int dt = (3.8\ mm)/c - \tau_{150}$. We must also note that although the area-entropy ratio converges to the appropriate value of $4\ell_p^2$, there is still a very slight systematic slope which causes the ratio to diverge. This is most likely due to an additional hard photon emission process which is occurring that is not described by our power spectrum, i.e. it has a sufficiently low emission rate that prohibits it from thermalizing. This is also reflected in the oscillation of the chi-squared at the higher frequencies.

In order to compute the Rindler bath temperature, we utilized the best fit parameters of the original theoretical power spectrum for each cutoff and applied them to the Rindler temperature-dependent power spectrum. We then performed best fits for each temperature at cutoffs from 30 GeV to 100 Gev in 10 GeV steps, and then computed the resulting chi-squares to ensure that each were still within the 1 standard deviation threshold. These best fit temperatures were then averaged to produce a Rindler bath temperature of $T_r = 1.96 \pm .49\ PeV$. When compared to the average FDU temperature $T_{FDU} = 1.80 \pm .51\ PeV$, we find the measured temperature of Rindler bath to be $T_r = T_{FDU}(1.09 \pm .41)$. The error comes from the standard deviation from the mean.

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B. The AQED Response Function

To begin our analysis we must first define the AQED response function. As we such, we examine electromagnetic emission in a refractive medium using the current interaction for QED

$$\hat{S}_I = \int d^4x \hat{j}_\mu(x)\hat{A}^\mu(x). \tag{S1}$$

We shall couple an Unruh-DeWitt detector to the vector current. This will endow the electron an extra degree of freedom for energy transitions, i.e. the recoil. As such,

$$\hat{j}_\mu(x) = u_\mu \hat{q}(\tau)\delta^3(x - x_{tr}(\tau)). \tag{S2}$$

The monopole moment operator $\hat{q}(t)$ is Heisenberg evolved via $\hat{q}(\tau) = e^{i\hat{H}\tau}\hat{q}(0)e^{-i\hat{H}\tau}$ with $\hat{q}(0)$ defined as $\hat{q}(0)|E_i\rangle = |E_f\rangle$ with E_i and E_f the initial energy and final energy of a two level system moving along the trajectory, $x_{tr}(t)$, of the current; transitions both up and down energy are allowed. With the intent to examine Larmor radiation, both in vacuum and an optical medium, we formulate the following amplitude;

$$\mathcal{A} = i\langle \mathbf{k}| \otimes \langle E_f| \hat{S}_i |E_i\rangle \otimes |0\rangle . \tag{S3}$$

The differential probability per unit final state momenta is given by, $\frac{d\mathcal{P}}{d^3k} = |\mathcal{A}|^2$. Evaluation yields

$$\begin{aligned}\frac{d\mathcal{P}}{d^3k} &= |\langle \mathbf{k}| \otimes \langle E_f| \int d^4x \hat{j}_\mu(x)\hat{A}^\mu(x)|E_i\rangle \otimes |0\rangle|^2 \\ &= \int d^4x \int d^4x' |\langle E_f|\hat{j}_\mu(x)|E_i\rangle|^2 |\langle \mathbf{k}|\hat{A}^\mu(x)|0\rangle|^2. \end{aligned}\tag{S4}$$

Note, the probability factorizes into an electron matrix element contracted with the photon matrix element. The electron matrix element yields

$$\begin{aligned}|\langle E_f|\hat{j}_\mu(x)|E_i\rangle|^2 &= |\langle E_f| u_\mu(x)e^{i\hat{H}\tau}\hat{q}(0)e^{-i\hat{H}\tau}\delta^3(x - x(\tau))|E_i\rangle|^2 \\ &= q^2 U_{\mu\nu}[x', x]\delta^3(x - x_{tr}(\tau))\delta^3(x' - x'_{tr}(\tau'))e^{-i\Delta E(\tau'-\tau)} \end{aligned}\tag{S5}$$

Here we have defined the energy gap as $\Delta E = E_f - E_i$ and the charge as $q^2 = |\langle E_f|\hat{q}(0)|E_i\rangle|^2$. For the sake of brevity we defined a "velocity tensor" via $U_{\mu\nu}[x', x] = u_\nu(x')u_\mu(x)$. Next, we shall evaluate the photon inner product. For this we will need to integrate over the final state momenta, thereby developing the total emission probability. Hence,

$$\begin{aligned}\int d^3k |\langle \mathbf{k}|\hat{A}^\mu(x)|0\rangle|^2 &= \int d^3k \langle 0|\hat{A}^{\dagger\nu}(x')|\mathbf{k}\rangle\langle \mathbf{k}|\hat{A}^\mu(x)|0\rangle \\ &= \langle 0|\hat{A}^{\dagger\nu}(x')\hat{A}^\mu(x)|0\rangle \\ &= G^{\nu\mu}[x', x]. \end{aligned}\tag{S6}$$

Note we have utilized the completeness relation, $\int dk |k\rangle\langle k| = 1$, to simplify the expression . The resultant is our photon two point function with vector indices. Using our photon two point function and the electron current density from Eq. (S5). we can formulate the AQED response function $\frac{d\mathcal{P}}{d\eta} = \Gamma$. Hence

$$\begin{aligned}\mathcal{P} &= \int d^3k \int d^4x \int d^4x' |\langle E_f|\hat{j}_\mu(x)|E_i\rangle|^2 |\langle \mathbf{k}|\hat{A}^\mu(x)|0\rangle|^2 \\ \Rightarrow \Gamma &= q^2 \int d\xi e^{-i\Delta E\xi} U_{\mu\nu}[x', x]G^{\nu\mu}[x', x]. \end{aligned}\tag{S7}$$

Here we have made use of the difference and average propertime change of variables; $\xi = \tau' - \tau$ and $\eta = (\tau' + \tau)/2$ respectively. Using the standard mode decomposition for the vector field in a dielectric medium, we have

$$\begin{aligned} \hat{A}^\mu(x) &= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\sum_i \epsilon_i^\mu}{\sqrt{2\omega}} \left[\hat{a}_k e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + \hat{a}_k^\dagger e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \right] \\ &= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\sigma^\mu}{\sqrt{2\omega}} \left[\hat{a}_k e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + \hat{a}_k^\dagger e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \right]. \end{aligned} \tag{S8}$$

In the last line we have defined the quantity $\sigma^\mu = \sum_i \epsilon_i^\mu$. The two point function then reduces to an integral over the momentum,

$$\begin{aligned} \langle 0 | \hat{A}^{\dagger\nu}(x') \hat{A}^\mu(x) | 0 \rangle &= \langle 0 | \int \frac{d^3k'}{(2\pi)^{3/2}} \frac{\sigma'^{\dagger\nu}}{\sqrt{2\omega'}} \left[\hat{a}_{k'} e^{i(\mathbf{k}'\cdot\mathbf{x}' - \omega' t')} + \hat{a}_{k'}^\dagger e^{-i(\mathbf{k}'\cdot\mathbf{x}' - \omega' t')} \right] \\ &\quad \times \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\sigma^\mu}{\sqrt{2\omega}} \left[\hat{a}_k e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + \hat{a}_k^\dagger e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \right] | 0 \rangle \\ &= \frac{1}{(2\pi)^3} \frac{1}{2} \int \frac{d^3k}{\omega} \sigma^\mu \sigma^{\dagger\nu} e^{i(\mathbf{k}\cdot\Delta\mathbf{x} - \omega(t' - t))}. \end{aligned} \tag{S9}$$

Again we see that the vector two point function is formally the same as a scalar field but with polarization vectors lending their indices. Combining all the pieces we can formulate the response function for our photon emission.

$$\Gamma = q^2 \frac{1}{(2\pi)^3} \frac{1}{2} \int d\xi \int \frac{d^3k}{\omega} U e^{-i(\Delta E \xi - \mathbf{k}\cdot\Delta\mathbf{x}_{tr} + \omega \Delta t)}. \tag{S10}$$

We defined the velocity product $U = \sigma^\mu \sigma^{\dagger\nu} U_{\mu\nu}[x', x]$ for brevity. Let us now examine the power radiated by a uniformly accelerated charge.

C. The Thermalized Larmor Formula

To analyze Larmor emission we will now consider the electron propagating in free space, i.e. $n = 1$. We will begin by examining the polarization vectors that are contracted with our velocity tensor. Recalling that under proper acceleration a, the four-velocities at proper time τ will be given $u^\mu = (\cosh(a\tau), 0, 0, \sinh(a\tau))$. Hence,

$$\begin{aligned} U &= \left(\sum_i u_\mu \epsilon_i^\mu \right) \left(\sum_j u_\nu \epsilon_j^\nu \right)^\dagger \\ &= \left(\sum_i \mathbf{u} \cdot \epsilon_i \right) \left(\sum_j \mathbf{u}' \cdot \epsilon_j \right)^\dagger \\ &= \sinh(a\tau) \sinh(a\tau') \sin^2(\theta). \end{aligned} \tag{S11}$$

Here θ is the angle of photon emission relative to the direction of propagation along the z-axis. Moreover we will make use of the hyperbolic double angle formulas to obtain

$$\sinh(a\tau) \sinh(a\tau') = \frac{1}{2} \left[2\cosh^2(a\eta) - 1 - \cosh(a\xi) \right]. \tag{S12}$$

Combining all the above pieces we can now formulate the transition probability. Thus

$$\Gamma = q^2 \frac{1}{(2\pi)^3} \frac{1}{4} \int d\xi \int \frac{d^3k}{\omega} \left[2\cosh^2(a\eta) - 1 - \cosh(a\xi) \right] \sin^2(\theta) e^{-i(\Delta E \xi - \mathbf{k}\cdot\Delta\mathbf{x}_{tr} + \omega \Delta t)}. \tag{S13}$$

To arrive at the Larmor formula, typically computed in the rest frame of the electron, we follow [53] and set $\beta = 0$; i.e. $\Delta x \ll \Delta t$. Note that with respect to the variable η, the Lorentz gamma is given by $\gamma = \cosh(a\eta)$ which we also take to be 1. As such we obtain

$$\Gamma = q^2 \frac{1}{(2\pi)^3} \frac{1}{4} \int d\xi \left[1 - \cosh(a\xi)\right] \int \frac{d^3k}{\omega} \sin^2(\theta) e^{-i(\Delta E \xi + \omega \Delta t)}. \tag{S14}$$

Now, examining the momentum integrations, we move to spherical coordinates with the momentum aligned along the z-axis, to yield

$$\begin{aligned}\Gamma &= q^2 \frac{1}{(2\pi)^3} \frac{1}{4} \int d\xi \left[1 - \cosh(a\xi)\right] \int d\theta d\omega d\phi \omega \sin^3(\theta) e^{-i(\Delta E \xi + \omega \Delta t)} \\ &= \frac{2}{3} \alpha \frac{1}{2\pi} \int d\xi \left[1 - \cosh(a\xi)\right] \int d\omega \omega e^{-i(\Delta E \xi + \omega \Delta t)}\end{aligned} \tag{S15}$$

Note, in the last line we rewrote the prefactor in terms of the fine structure constant $\alpha = \frac{q^2}{4\pi}$.To compute the Larmor formula, we will need to examine the power emitted by the photon. As such we weight the frequency integral with an additional factor of frequency. Hence

$$\mathcal{S} = \frac{2}{3} \alpha \frac{1}{2\pi} \int d\xi e^{-i\Delta E \xi} \left[1 - \cosh(a\xi)\right] \int d\omega \omega^2 e^{-i\omega \Delta t}. \tag{S16}$$

The integration over the frequency can now be carried out to yield

$$\int d\omega \omega^2 e^{-i\omega \Delta t} = \frac{2i}{\Delta t^3}. \tag{S17}$$

We should note that there is an implicit regulator $\Delta t \to \Delta t - i\epsilon$ in the denominator. This will later require us to include a pole on the real axis in the integration over the proper time. Our integration now simplifies to

$$\mathcal{S} = \frac{2}{3} \alpha \frac{i}{\pi} \int d\xi e^{-i\Delta E \xi} \frac{\left[1 - \cosh(a\xi)\right]}{\Delta t^3}. \tag{S18}$$

Finally we recall that $\Delta t = \frac{2}{a} \sinh(a\xi/2)$, we have

$$\mathcal{S} = \frac{2}{3} \alpha \frac{i}{\pi} \left(\frac{a}{2}\right)^3 \int d\xi e^{-i\Delta E \xi} \frac{\left[1 - \cosh(a\xi)\right]}{\sinh^3(a\xi/2)}. \tag{S19}$$

Converting the hyperbolic terms to exponentials and making the change of variables $w = e^{a\xi}$, we have

$$\mathcal{S} = \frac{2}{3} \alpha \frac{i}{\pi} \left(\frac{a}{2}\right)^3 \frac{8}{a} \int dw \frac{\left[w^{1/2 - i\Delta E/a} - \frac{1}{2} w^{3/2 - i\Delta E/a} - \frac{1}{2} w^{-1/2 - i\Delta E/a}\right]}{\left[w - 1\right]^3}. \tag{S20}$$

This integration is standard and can be evaluated using the residue theorem. As such we obtain the following

$$\mathcal{S} = \frac{2}{3} \alpha a^2 \frac{1}{1 + e^{2\pi \Delta E/a}}. \tag{S21}$$

This is our thermal Larmor formula. By summing over transitions both up and down in energy, i.e. when $\Delta E = |\Delta E|$ and $\Delta E = -|\Delta E|$, and taking the limit $|\Delta E| \to 0$ we arrive at the standard Larmor formula. Hence

$$\mathcal{S} = \frac{2}{3} \alpha a^2. \tag{S22}$$

Note, this is written in terms of the proper acceleration and therefore is fully relativistic as in the classical derivation.

D. The Quantum Correction to the Larmor Formula

The quantum correction to the Larmor formula will come from the recoil correction. Generically, by setting the energy gap to $\frac{\omega^2}{2m}$ we can examine what the quantum correction will be. Let us consider Eq. (S15) of the supplementary.

$$\Gamma = \frac{2}{3}\alpha\frac{1}{2\pi}\int d\xi \left[1-\cosh\left(a\xi\right)\right]\int d\omega\omega e^{-i(\Delta E\xi+\omega\Delta t)} \tag{S23}$$

Defining $\Delta\bar{E}$ as the auxiliary gap, the total energy gap we will use is defined as follows;

$$\Delta E = \frac{\omega^2}{2m}+\Delta\bar{E}. \tag{S24}$$

With this energy gap we note that the recoil portion must be integrated over with the frequency while the auxiliary gap serves the same role as a traditional Unruh-DeWitt detector. As such we split the detector up into the two portions; the frequency dependent part and the auxiliary gap. Thus,

$$\begin{aligned}\Gamma &= \frac{2}{3}\alpha\frac{1}{2\pi}\int d\xi \left[1-\cosh\left(a\xi\right)\right]\int d\omega\omega e^{-i(\Delta E\xi+\omega\Delta t)}\\ &= \frac{2}{3}\alpha\frac{1}{2\pi}\int d\xi e^{-i\Delta\bar{E}\xi} \left[1-\cosh\left(a\xi\right)\right]\int d\omega\omega e^{-i(\frac{\omega^2}{2m}\xi+\omega\Delta t)}.\end{aligned} \tag{S25}$$

To compute the Larmor formula, we will need to examine the power emitted by the photon. As such we weight the frequency integral with an additional factor of frequency. Hence

$$\mathcal{S} = \frac{2}{3}\alpha\frac{1}{2\pi}\int d\xi e^{-i\Delta\bar{E}\xi} \left[1-\cosh\left(a\xi\right)\right]\int d\omega\omega^2 e^{-i(\frac{\omega^2}{2m}\Delta t+\omega\Delta t)}. \tag{S26}$$

Note that since we are in a non relativistic regime we can make the replacement $\xi\to\Delta t$ in the frequency integration. Moreover, to evaluate the resultant integrals we Taylor expand the exponent that is quadratic in the frequency to yield

$$\mathcal{S} = \frac{2}{3}\alpha\frac{1}{2\pi}\int d\xi e^{-i\Delta\bar{E}\xi} \left[1-\cosh\left(a\xi\right)\right]\sum_{\ell=0}^{\infty}\frac{\left(\frac{-i\Delta t}{2m}\right)^{\ell}}{\ell!}\int d\omega\omega^{2(\ell+1)}e^{-i\omega\Delta t}. \tag{S27}$$

The integrations over the frequency can now be carried out using the identity

$$\int d\omega\omega^{2(\ell+1)}e^{-i\omega\Delta t} = \frac{(-1)^{2\ell+3}(2\ell+2)!}{(-i\Delta t)^{2\ell+3}}. \tag{S28}$$

We should note that there is an implicit regulator $\Delta t\to\Delta t-i\epsilon$ in the denominator. This will later require us to include a pole on the real axis in the integration over the proper time. Our integration now simplifies to

$$\begin{aligned}\mathcal{S} &= \frac{2}{3}\alpha\frac{1}{2\pi}\int d\xi e^{-i\Delta\bar{E}\xi} \left[1-\cosh\left(a\xi\right)\right]\sum_{\ell=0}^{\infty}\frac{\left(\frac{-i\Delta t}{2m}\right)^{\ell}}{\ell!}\int d\omega\omega^{2(\ell+1)}e^{-i\omega\Delta t}\\ &= \frac{2}{3}\alpha\frac{1}{2\pi}\sum_{\ell=0}^{\infty}\frac{(2\ell+2)!}{(-2m)^{\ell}\ell!}\frac{1}{(i)^{\ell+3}}\int d\xi\frac{e^{-i\Delta\bar{E}\xi}\left[1-\cosh\left(a\xi\right)\right]}{(i\Delta T)^{\ell+3}}.\end{aligned} \tag{S29}$$

Finally we recall that $\Delta t = \frac{2}{a}\sinh\left(a\xi/2\right)$, we have

$$\mathcal{S} = \frac{2}{3}\alpha\frac{1}{2\pi}\sum_{\ell=0}^{\infty}\frac{(2\ell+2)!}{(-2m)^{\ell}\ell!}\left(\frac{a}{2i}\right)^{\ell+3}\int d\xi\frac{e^{-i\Delta\bar{E}\xi}\left[1-\cosh\left(a\xi\right)\right]}{\sinh^{\ell+3}\left(a\xi/2\right)}. \tag{S30}$$

These integrations are standard and can be evaluated on a case by case basis. For the case of $\ell = 0$ we obtain the following

$$\mathcal{S}_0 = \frac{2}{3}\alpha a^2 \frac{1}{1 + e^{2\pi\Delta\bar{E}/a}}. \tag{S31}$$

By summing over transition up and down in energy, i.e. when $\Delta\bar{E} = |\Delta\bar{E}|$ and $\Delta\bar{E} = -|\Delta\bar{E}|$, and taking the limit $|\Delta\bar{E}| \to 0$ we arrive at the Larmor formula. Hence

$$\mathcal{S}_0 = \frac{2}{3}\alpha a^2. \tag{S32}$$

For the first quantum correction to Larmor we evaluate the $\ell = 1$ integral. As such we obtain

$$\mathcal{S}_1 = \frac{4\alpha}{m} a^2 \frac{\Delta\bar{E}}{1 - e^{2\pi\Delta\bar{E}/a}}. \tag{S33}$$

Then by summing over transition up and down in energy we obtain

$$\begin{aligned} \mathcal{S}_1 &= \frac{4\alpha}{m} a^2 \Delta\bar{E} \frac{\sinh\left(2\pi\Delta\bar{E}/a\right)}{1 - \cosh\left(2\pi\Delta\bar{E}/a\right)} \\ &= -\frac{4\alpha}{\pi m} a^3 \\ &= -\frac{8\alpha}{m} a^2 T_{FDU}. \end{aligned} \tag{S34}$$

Note, in the last line we took the limit of zero auxiliary energy gap. This is the quantum correction to the Larmor formula. Next we compute the radiation reaction force for each term.

$$\begin{aligned} \int F^{rr} v \cdot dt &= -\int \mathcal{S} dt \\ \Rightarrow F_0^{rr} &= \frac{2}{3}\alpha J \\ \Rightarrow F_1^{rr} &= -\frac{8\alpha}{\pi m} J a. \end{aligned} \tag{S35}$$

As such we find the covariant form of the quantum LAD equation to be

$$m\frac{du^\mu}{ds} = qF^{\mu\nu}u_\nu + \frac{2}{3}\alpha\hbar\left(1 - \frac{24\hbar}{\pi m c^3}\sqrt{a^2}\right)\left[J^\mu + a^2 u^\mu\right]. \tag{S36}$$

E. Power Spectrum

Prior to integrating over the emitted photons frequency, Eq. (S14), we will have the following emission rate

$$\Gamma = q^2 \frac{1}{(2\pi)^3}\frac{1}{4}\int d\xi \int \frac{d^3k}{\omega}\left[2\cosh^2(a\eta) - 1 - \cosh(a\xi)\right]\sin^2(\theta) e^{-i(\Delta E\xi - \mathbf{k}\cdot\Delta\mathbf{x}_{tr} + \omega\Delta t)}. \tag{S37}$$

We will further simplify by using the following redefinition $\delta = 2\gamma^2 - 1$. Utilizing the same approximation $\Delta x \sim 0$, which reproduces the Larmor formula, but now keeping our boost parameter, which is a function of η only, we can now convert to spherical coordinates and integrate over the emission angle. Hence

$$\Gamma = q^2 \frac{1}{(2\pi)^3}\frac{1}{4}\int d\xi \int \frac{d^3k}{\omega}\left[\delta - \cosh\left(a\xi\right)\right]\sin^2\left(\theta\right)e^{-i(\Delta E\xi + \omega\Delta t)}$$
$$= q^2 \frac{1}{(2\pi)^2}\frac{1}{3}\int d\xi \int d\omega\omega\left[\delta - \cosh\left(a\xi\right)\right]e^{-i(\Delta E\xi + \omega\Delta t)} \tag{S38}$$

weighting by an extra factor of frequency to obtain the power, we formulate the power spectrum, $\frac{d\mathcal{S}}{d\omega}$. Hence

$$\frac{d\mathcal{S}}{d\omega} = q^2 \frac{1}{(2\pi)^2}\frac{1}{3}\int d\xi\omega^2\left[\delta - \cosh\left(a\xi\right)\right]e^{-i(\Delta E\xi + \omega\Delta t)} \tag{S39}$$

Recalling that with the boost parameter, $\Delta t = \frac{2}{a}\sinh\left(a\xi/2\right)\gamma$, we now convert hyperbolic cosine to exponentials to obtain

$$\frac{d\mathcal{S}}{d\omega} = q^2 \frac{1}{(2\pi)^2}\frac{\omega^2}{3}\int d\xi \Bigg[\ \delta e^{-i(\Delta E\xi + \frac{2\omega\gamma}{a}\sinh\left(a\xi/2\right))} - \frac{1}{2}e^{-i((\Delta E + ia)\xi + \frac{2\omega\gamma}{a}\sinh\left(a\xi/2\right))} - \frac{1}{2}e^{-i((\Delta E - ia)\xi + \frac{2\omega\gamma}{a}\sinh\left(a\xi/2\right))}\Bigg]. \tag{S40}$$

Employing the change of variables $w = a\xi/2$ we obtain

$$\frac{d\mathcal{S}}{d\omega} = q^2 \frac{1}{(2\pi)^2}\frac{\omega^2}{3}\frac{2}{a}\int dw \Bigg[\ \delta e^{(-\frac{2i\Delta E}{a}w - \frac{2i\omega\gamma}{a}\sinh\left(w\right))} - \frac{1}{2}e^{(-(\frac{2i\Delta E}{a}-2)w - \frac{2i\omega\gamma}{a}\sinh\left(w\right))} - \frac{1}{2}e^{(-(\frac{2i\Delta E}{a}+2)w - \frac{2i\omega\gamma}{a}\sinh\left(w\right))}\Bigg]. \tag{S41}$$

Now, recalling the integral representation of the second Hankel function, we have

$$H_n^{(2)}(x) = -\frac{1}{i\pi}\int_{-\infty}^{\infty - i\pi} dt e^{-nt + x\sinh(t)}. \tag{S42}$$

Here, the integration contour is shifted down by $-\pi$ on the imaginary axis. This is consistent with the Larmor case since there we used our $\Delta t - i\epsilon$ prescription. Using the above formula we find our power spectrum to be

$$\frac{d\mathcal{S}}{d\omega} = -i\frac{2}{3}\alpha\frac{\omega^2}{a}\left[\delta H^{(2)}_{\frac{2i\Delta E}{a}}\left(-\frac{2i\omega\gamma}{a}\right) - \frac{1}{2}\left(H^{(2)}_{\frac{2i\Delta E}{a}-2}\left(-\frac{2i\omega\gamma}{a}\right) + H^{(2)}_{\frac{2i\Delta E}{a}+2}\left(-\frac{2i\omega\gamma}{a}\right)\right)\right]. \tag{S43}$$

To make the presence of thermality more apparent, we make use of the following identity, $H_\ell^{(2)}(x) = e^{i\ell\pi}H_{-\ell}^{(2)}(x)$. As such, each term will yield precisely a Boltzmann factor with the Unruh-DeWitt detector energy gap thermalized at the celebrated Fulling-Davies-Unruh temperature. Applying this identity then yields our power spectrum,

$$\frac{d\mathcal{S}}{d\omega} = -i\frac{2}{3}\alpha\frac{\omega^2}{a}e^{-2\pi\Delta E/a}\left[\delta H^{(2)}_{-\frac{2i\Delta E}{a}}\left(-\frac{2i\omega\gamma}{a}\right) - \frac{1}{2}\left(H^{(2)}_{-\frac{2i\Delta E}{a}+2}\left(-\frac{2i\omega\gamma}{a}\right) + H^{(2)}_{\frac{-2i\Delta E}{a}-2}\left(-\frac{2i\omega\gamma}{a}\right)\right)\right]. \tag{S44}$$

This power spectrum contains the Unruh-DeWitt detector energy gap thermalized at the Fulling-Davies-Unruh temperature. Let us now explore the use of radiative energy loss, as a source for acceleration. We will then compare our results with the recent experimental observation of radiation reaction in aligned crystals.

F. The Thermalized Total Transverse Emission Spectrum

In order to understand the physical processes present in the Rindler frame we must first start by computing the emission spectrum per unit transverse momentum, i.e. $\frac{d\Gamma}{dk_\perp}$. In the Rindler frame, the mode functions of the photons are labeled by momentum transverse to the acceleration and by the Rindler frequency ω_r which characterizes the photons energy in Rindler frame. We will find that ω_r will play the role of the energy gap of the Unruh-DeWitt detector. Moreover, there is no dispersion relation which relates ω_r to $k_\perp$. For this reason, we must utilize our formalism to compute $\frac{d\Gamma}{d^2k_\perp}$ in both the Rindler frame and the lab frame to confirm that they match. Starting with the lab frame computation, let us recall Eq. (S37),

$$\begin{aligned}\Gamma &= q^2\frac{1}{(2\pi)^3}\frac{1}{4}\int d\xi\int\frac{d^3k}{\omega}\left[2\cosh^2(a\eta)-1-\cosh(a\xi)\right]\sin^2(\theta)e^{-i(\Delta E\xi-\mathbf{k}\cdot\Delta\mathbf{x}_{tr}+\omega\Delta t)}\\ \frac{d\Gamma}{d^2k_\perp} &= q^2\frac{1}{(2\pi)^3}\frac{1}{4}\int d\xi\int\frac{dk_z}{\omega}\left[\delta-\cosh(a\xi)\right]\sin^2(\theta)e^{-i(\Delta E\xi-\mathbf{k}\cdot\Delta\mathbf{x}_{tr}+\omega\Delta t)}\end{aligned} \tag{S45}$$

Note again we made use of the boost factor $\delta=2\cosh^2(a\eta)-1$ and we have defined $d^2k_\perp=dk_xdk_y$. Using the same approximation $\Delta x\sim 0$ and recalling $\Delta t=\frac{2}{a}\sinh(a\xi/2)\gamma$, we have

$$\frac{d\Gamma}{d^2k_\perp}=q^2\frac{1}{(2\pi)^3}\frac{1}{4}\int d\xi\int\frac{dk_z}{\omega}\left[\delta-\cosh(a\xi)\right]\sin^2(\theta)e^{-i(\Delta E\xi+\frac{2\omega\gamma}{a}\sinh(a\xi/2))}. \tag{S46}$$

Utilizing the same Hankel identity, Eq. (S42), and recalling $\sin(\theta)=\frac{k_\perp}{\omega}$ we arrive at,

$$\begin{aligned}\frac{d\Gamma}{d^2k_\perp} =& \frac{-i\alpha}{4\pi a}\int dk_z\frac{k_\perp^2}{\omega^3}\Bigg[\delta H^{(2)}_{\frac{2i\Delta E}{a}}\left(-\frac{2i\omega\gamma}{a}\right)\\ &-\frac{1}{2}\left(H^{(2)}_{\frac{2i\Delta E}{a}-2}\left(-\frac{2i\omega\gamma}{a}\right)+H^{(2)}_{\frac{2i\Delta E}{a}+2}\left(-\frac{2i\omega\gamma}{a}\right)\right)\Bigg].\end{aligned} \tag{S47}$$

Finally, if we recall that $\omega=\sqrt{k_\perp^2+k_z^2}$ and assume that the functional dependence on k_z in the argument of the Hankel functions is negligible, i.e. $\omega\sim k_\perp$, then the integral over k_z can be evaluated. Since each photon from a thermalized process will be emitted into Rindler horizon. This occurs due to diphoton creation near the Rindler horizon which causes one to escape into the horizon, while the other is absorbed by the electron or by direct photon emission into the horizon; i.e. emission and absorption of Rindler photons [24, 52]. This implies the momentum of all thermalized photons, in this setting, will have $k_z\geq 0$. This fact may be of pertinence to understanding the nature of the relationship between the Minkowski photon energy ω and the Rindler photon frequency ω_r. Hence

$$\begin{aligned}\int dk_z\frac{k_\perp^2}{\omega^3} &= \int_0^\infty dk_z\frac{k_\perp^2}{(k_\perp^2+k_z^2)^{3/2}}\\ &=1.\end{aligned} \tag{S48}$$

From here, we obtain our emission rate per unit transverse momentum,

$$\begin{aligned}\frac{d\Gamma}{d^2k_\perp} =& \frac{-i\alpha}{4\pi a}\Bigg[\delta H^{(2)}_{\frac{2i\Delta E}{a}}\left(-\frac{2ik_\perp\gamma}{a}\right)\\ &-\frac{1}{2}\left(H^{(2)}_{\frac{2i\Delta E}{a}-2}\left(-\frac{2ik_\perp\gamma}{a}\right)+H^{(2)}_{\frac{2i\Delta E}{a}+2}\left(-\frac{2ik_\perp\gamma}{a}\right)\right)\Bigg].\end{aligned} \tag{S49}$$

Finally, we recall that each Hankel function, by flipping the sign in the index, produces the detailed balance relationship. From this, when we can form the total emission rate, $\frac{d\Gamma_{tot}}{d^2k_\perp}=\frac{d\Gamma_{\Delta E}}{d^2k_\perp}+\frac{d\Gamma_{-\Delta E}}{d^2k_\perp}$. This yields,

$$\begin{aligned}\frac{d\Gamma_{tot}}{d^2k_\perp} &= \frac{d\Gamma_{\Delta E}}{d^2k_\perp} + \frac{d\Gamma_{-\Delta E}}{d^2k_\perp} \\ &= \frac{d\Gamma_{\Delta E}}{d^2k_\perp} + e^{2\pi\Delta E/a}\frac{d\Gamma_{\Delta E}}{d^2k_\perp} \\ &= \frac{d\Gamma}{d^2k_\perp}\left[1+e^{2\pi\Delta E/a}\right]. \end{aligned} \tag{S50}$$

Here, $\frac{d\Gamma}{d^2k_\perp}$, is given by Eq. (S47). Writing it explicitely for later comparion we find,

$$\begin{aligned}\frac{d\Gamma_{tot}}{d^2k_\perp} = \frac{-i\alpha}{4\pi a}&\left[\delta H^{(2)}_{\frac{2i\Delta E}{a}}\left(-\frac{2ik_\perp\gamma}{a}\right) - \frac{1}{2}\left(H^{(2)}_{\frac{2i\Delta E}{a}-2}\left(-\frac{2ik_\perp\gamma}{a}\right) + H^{(2)}_{\frac{2i\Delta E}{a}+2}\left(-\frac{2ik_\perp\gamma}{a}\right)\right)\right] \\ &\times\left[1+e^{2\pi\Delta E/a}\right]. \end{aligned} \tag{S51}$$

Note, that this also assumes that the energy gap, if dependent on the photon frequency ω, also reduces to $\Delta E(\omega) \to \Delta E(k_\perp)$. We must now ensure that the channeling radiation analysis, in the Rindler frame, matches the above expression.

G. The Rindler Frame Analysis of a "Channeling like" Oscillation

The Rindler frame analysis provides us with a special opportunity to explore the particle content of the background spacetime and potential temperature experienced by an accelerated particle. In order to approach this computation, we must first look at our Rindler line element for the coordinates (τ, ξ, x, y); $ds^2 = e^{2a\xi}\left(d\tau^2 - d\xi^2\right) - dx_\perp^2$. Here, ξ characterizes motion along the z direction, τ is the Rindler time, and $x_\perp^2 = x^2 + y^2$. The two Rindler coordinates (τ, ξ) are related to the laboratory time and z cooridinate via; $t = (e^{a\xi}/a)\sinh(a\tau)$ and $z = (e^{a\xi}/a)\cosh(a\tau)$. Quantization of the electromagnetic field in the Rindler wedge yields the two physical photon modes,

$$\begin{aligned} A^1_\mu(x) &= \frac{1}{2\pi^2 k_\perp}\left(\frac{\sinh(\pi\omega_r/a)}{a}\right)^{1/2}(0,0,k_y f(x), -k_x f(x)) \\ A^2_\mu(x) &= \frac{1}{2\pi^2 k_\perp}\left(\frac{\sinh(\pi\omega_r/a)}{a}\right)^{1/2}(\partial_\xi f(x), -i\omega_r f(x), 0, 0). \end{aligned} \tag{S52}$$

The transverse photon momentum k_x and k_y along with the mutually independent Rindler frequency ω_r label the Rindler photon mode functions; here $k_\perp = \sqrt{k_x^2 + k_y^2}$. The function $f(x)$ which characterizes the spatial modulation of the modes is given by,

$$f(x) = K_{i\omega_r/a}\left(\frac{k_\perp}{a}e^{a\xi}\right)e^{i(k_\perp\cdot x_\perp - \omega_r\tau)}. \tag{S53}$$

Our second quantized field operators are constructed from the above physics polarization modes, integrated over the transverse momentum and Rindler frequency,

$$\hat{A}_\mu(x) = \int_\infty^\infty d^2k_\perp \int_0^\infty d\omega_r \sum_{i=1,2}\left[\hat{a}A^i_\mu(x) + \hat{a}^\dagger A^{i\dagger}_\mu(x)\right]. \tag{S54}$$

Here we are summing over both physical polarization modes. Now, in order to analyze any photon emission or absorption processes, we will use the curved spacetime QED interaction Lagrangian, $\mathcal{L} = \sqrt{-g}j^\mu_r\hat{A}_m$. Here, the Rindler current will characterize the comoving channeling oscillation. The resulting amplitude $\mathcal{A}$ for Rindler photon absorption is given by

$$\mathcal{A}_{abs} = i\int d^4x\sqrt{-g}j^\mu_r\langle 0|\hat{A}_\mu|\gamma\rangle. \tag{S55}$$

The magnitude for Rindler photon emission is the same for absorption, i.e. $|\mathcal{A}_{abs}| = |\mathcal{A}_{emi}|$. Moreover, if we consider the presence of a thermal backround in Rindler space, when we compute the total probability for both photon emission and absorption, we must take into account the background thermal bath. What this means is that when we compute the total probability, we must weight each component by the contribution from the thermal background, i.e. $\mathcal{P}_{abs} \sim |\mathcal{A}_{abs}|^2(1/e^{\omega_r/T}-1)$ and $\mathcal{P}_{emi} \sim |\mathcal{A}_{abs}|^2(1+1/(e^{\omega_r/T}-1))$. The temperature of the background thermal bath, T, is kept arbitrary. Summing both probabilities and integrating over the final Rindler photon states then gives the total probability,

$$\mathcal{P} = \sum_{i=1,2}\int_{\infty}^{\infty} d^2k_\perp \int_0^\infty d\omega_r |\mathcal{A}^i_{abs}|^2 \coth{(\omega_r/(2T))}. \tag{S56}$$

Now, in order to evaluate our absorption amplitude, we need the functional form of the Rindler current. The four velocity for a channeling like oscillation, e.g. a transverse oscillation in the x direction, can be written as,

$$u^\mu = \gamma_x(1, 0, v_0\cos{(\Omega\tau)}, 0). \tag{S57}$$

Here the velocity parameter, $v_0 = A\Omega$, is determined the oscillation amplitude, A, and oscillation frequency, Ω. The Lorentz factor is determined from the transverse oscillation which we take to be non relativistic, i.e. $\gamma_x = 1$. From here we can construct our four current for the channeling oscillation is then given by

$$j^\mu_r = qu^\mu \delta(\xi)\delta(x - x_{tr})\delta(y). \tag{S58}$$

The trajectory along the x direction is given by $x = A\sin{(\Omega\tau)}$ and q is the electron charge. Utilizing this trajectory, we can examine the total rate of Rindler photon emission and absorption under the assumption that there is a background thermal bath present in the Rindler frame. Note, for the transverse oscillation, we will see that it is sufficient to only include the minimal oscillation term in the phase and keep the velocity term in the current constant, i.e. $v_x = v_0\cos{(\Omega\tau)} \sim v_0$ and $x_{tr} = A\sin{(\Omega\tau)} \sim v_0\tau$. We also must recall the covariant volume element is given by $\sqrt{-g} = e^{2a\xi}$. Recalling Eq. (S55), let us compute the absorption amplitude for the first polarization. Hence,

$$\begin{aligned}
\mathcal{A}^1_{abs} &= i\int d^4x\sqrt{-g}j^\mu_r \langle 0|\, \hat{A}^1_\mu \,|\gamma\rangle \\
&= i\int d^4x\sqrt{-g}j^\mu_r \int d^2k_\perp \int d\omega \,\langle 0|\left[\hat{a}A^1_\mu(x) + \hat{a}^\dagger A^{1\dagger}_\mu(x)\right]|\gamma\rangle \\
&= i\int d^4x\sqrt{-g}j^\mu_r A^1_\mu(x) \\
&= i\int d^4x e^{2a\xi}\delta(\xi)\delta(x-x_{tr})\delta(y)u^\mu A^1_\mu(x) \\
&= \frac{-iq}{2\pi^2}\int d^4x e^{2a\xi}\delta(\xi)\delta(x-x_{tr})\delta(y)\frac{v_0k_y}{k_\perp}\left(\frac{\sinh{(\pi\omega_r/a)}}{a}\right)^{1/2} K_{i\omega_r/a}\left(\frac{k_\perp}{a}e^{a\xi}\right)e^{i(k_\perp\cdot x_\perp - \omega_r\tau)} \\
&= \frac{-iq}{2\pi^2}\int d\tau \frac{v_0k_y}{k_\perp}\left(\frac{\sinh{(\pi\omega_r/a)}}{a}\right)^{1/2} K_{i\omega_r/a}\left(\frac{k_\perp}{a}\right)e^{i(k_xv_0-\omega_r)\tau} \\
&= \frac{-iq}{\pi}\frac{v_0k_y}{k_\perp}\left(\frac{\sinh{(\pi\omega_r/a)}}{a}\right)^{1/2} K_{i\omega_r/a}\left(\frac{k_\perp}{a}\right)\delta(\omega_r - k_xv_0).
\end{aligned} \tag{S59}$$

Similarly, for the second physical mode we obtain,

$$\begin{aligned}
\mathcal{A}^2_{abs} &= i\int d^4x\sqrt{-g}j^\mu_r A^2_\mu(x) \\
&= \frac{iq}{\pi}\left(\frac{\sinh{(\pi\omega_r/a)}}{a}\right)^{1/2} K'_{i\omega_r/a}\left(\frac{k_\perp}{a}\right)\delta(\omega_r - k_xv_0).
\end{aligned} \tag{S60}$$

Here, the derivative of the Bessel function is with respect to the argument. Now, by taking the magnitude squared of the above amplitudes, we can construct the total probability,

$$\begin{aligned}
\mathcal{P} &= \int_{\infty}^{\infty} d^2k_\perp \int_0^\infty d\omega_r \left[|\mathcal{A}^1_{abs}|^2 + |\mathcal{A}^2_{abs}|^2\right]\coth{(\omega_r/(2T))} \\
&= \frac{q^2}{\pi^2 a}\int_{\infty}^{\infty} d^2k_\perp \int_0^\infty d\omega_r \sinh{(\pi\omega_r/a)}\delta^2(\omega_r - k_xv_0)\coth{(\omega_r/(2T))} \\
&\times \left[\left(\frac{v_0k_y}{k_\perp}\right)^2 |K_{i\omega_r/a}\left(\frac{k_\perp}{a}\right)|^2 + |K'_{i\omega_r/a}\left(\frac{k_\perp}{a}\right)|^2\right].
\end{aligned} \tag{S61}$$

We can now convert one of the delta functions into a total interaction time via $\Delta\tau = 2\pi\delta(0)$. This will allow us to formulate the emission rate, $\Gamma = \mathcal{P}/\Delta\tau$. Then, after integrating over the remaining delta function we fix the Rindler frequency to the channeling oscillation, $\omega_r = k_x A\Omega$. We can now formulate the emission rate per transverse momentum $\frac{d\Gamma}{d^2k_\perp} =$. Hence,

$$\frac{d\Gamma}{d^2k_\perp} = \frac{q^2}{2\pi^3 a}\sinh(\pi\omega_r/a)\coth(\omega_r/(2T))\left[\left(\frac{v_0 k_y}{k_\perp}\right)^2 |K_{i\omega_r/a}\left(\frac{k_\perp}{a}\right)|^2 + |K'_{i\omega_r/a}\left(\frac{k_\perp}{a}\right)|^2\right]. \quad \text{(S62)}$$

We must note at this point that the integral over k_x goes from $-\infty \to \infty$. This is because although we only kept the minimal contribution of the oscillation in the exponent, $sin(\Omega\tau) \to \Omega\tau$, the velocity still takes negative values in practice so we keep the full integration. Let us now consider the transformation of the K Bessel functions into Hankel functions. First, we must recall the identity $K_\alpha(x) = \frac{\pi}{2}(-i)^{\alpha+1}H^{(2)}_\alpha(-ix)$. This identity allows us to write

$$\begin{aligned} K_{i\omega_r/a}\left(\frac{k_\perp}{a}\right) &= -i\frac{\pi}{2}e^{\frac{\pi\omega_r}{2a}} H^{(2)}_{i\omega_r/a}\left(-i\frac{k_\perp}{a}\right) \\ K'_{i\omega_r/a}\left(\frac{k_\perp}{a}\right) &= -i\frac{\pi}{2}e^{\frac{\pi\omega_r}{2a}} H^{(2)'}_{i\omega_r/a}\left(-i\frac{k_\perp}{a}\right). \end{aligned} \quad \text{(S63)}$$

From here, our emission spectrum transforms into,

$$\begin{aligned} \frac{d\Gamma}{d^2k_\perp} &= \frac{q^2}{8\pi a}\sinh(\pi\omega_r/a)\coth(\omega_r/(2T))e^{\frac{\pi\omega_r}{a}} \\ &\times \left[\left(\frac{v_0 k_y}{k_\perp}\right)^2 |H^{(2)}_{i\omega_r/a}\left(-i\frac{k_\perp}{a}\right)|^2 + |H^{(2)'}_{i\omega_r/a}\left(-i\frac{k_\perp}{a}\right)|^2\right]. \end{aligned} \quad \text{(S64)}$$

We can further reduce the above Hankel functions by recalling their integral identity, Eq. (S42). Let us consider the first Hankel term in the above expression

$$\begin{aligned} |H^{(2)}_{i\alpha}(-ix)|^2 &= \frac{1}{-i\pi}\int dt e^{i\alpha t - ix\sinh(t)}\frac{1}{i\pi}\int dt' e^{-i\alpha t' + ix\sinh(t')} \\ &= \frac{1}{\pi^2}\int dt \int dt' e^{i\alpha t - ix\sinh(t) - i\alpha t' + ix\sinh(t')} \\ &= \frac{1}{\pi^2}\int dt \int dt' e^{-i\alpha(t'-t) - ix(\sinh(t') - \sinh(t))} \\ &= \frac{2}{\pi^2}\int d\eta \int d\xi e^{-2i\alpha\xi - 2ix\sinh(\xi)\cosh(\eta)} \\ &= \frac{-2i}{\pi}\int d\eta \int H^{(2)}_{2i\alpha}(-i2x\gamma). \end{aligned} \quad \text{(S65)}$$

Here we made use of the change of variables $\xi = (t'-t)/2$ and $\eta = (t'+t)/2$. Moreover, note that we have recovered a Lorentz boost factor, $\gamma = \cosh(\eta)$ based on the rapidity variable η. Let us now evaluate the derivative term. To begin, we will transform the derivative back into pure Hankel functions via the use of the identity $H^{(2)'}_{i\alpha}(-ix) = -\frac{i}{2}\left[H^{(2)}_{i\alpha-1}(-ix) - H^{(2)}_{i\alpha+1}(-ix)\right]$. This will yield the intermediate step,

$$\begin{aligned} |H^{(2)'}_{i\alpha}(-ix)|^2 = \frac{1}{4}[&|H^{(2)}_{i\alpha-1}(-ix)|^2 + |H^{(2)}_{i\alpha+1}(-ix)|^2 \\ &- H^{(2)}_{i\alpha-1}(-ix)H^{(2)*}_{i\alpha+1}(-ix) - H^{(2)}_{i\alpha+1}(-ix)H^{(2)*}_{i\alpha-1}(-ix)] \end{aligned} \quad \text{(S66)}$$

Lets begin by evaluating the negative terms first. The first one reduces to,

$$\begin{aligned}
H^{(2)}_{i\alpha+1}(-ix)H^{(2)*}_{i\alpha-1}(-ix) &= \frac{1}{-i\pi}\int dt' e^{-(i\alpha-1)t'-ix\sinh(t')}\frac{1}{i\pi}\int dt e^{-(-i\alpha+1)t+ix\sinh(t)} \\
&= \frac{1}{\pi^2}\int dt\int dt' e^{-(i\alpha-1)t'-ix\sinh(t')-(-i\alpha+1)t+ix\sinh(t)} \\
&= \frac{1}{\pi^2}\int dt\int dt' e^{-i\alpha t'-t'-ix\sinh(t')+i\alpha t+t+ix\sinh(t)} \\
&= \frac{2}{\pi^2}\int d\eta\int d\xi e^{-2(i\alpha+1)\xi-2ix\sinh(\xi)\cosh(\eta)} \\
&= \frac{-2i}{\pi}\int d\eta H^{(2)}_{2i\alpha+2}(-i2x\gamma).
\end{aligned} \tag{S67}$$

Similary, evaluation of the other negative term yields,

$$\begin{aligned}
H^{(2)}_{i\alpha-1}(-ix)H^{(2)*}_{i\alpha+1}(-ix) &= \frac{1}{-i\pi}\int dt' e^{-(i\alpha-1)t'-ix\sinh(t')}\frac{1}{i\pi}\int dt e^{-(-i\alpha+1)t+ix\sinh(t)} \\
&= \frac{1}{\pi^2}\int dt'\int dt e^{-i\alpha t'+t'-ix\sinh(t')+i\alpha t-t+ix\sinh(t)} \\
&= \frac{1}{\pi^2}\int dt'\int dt e^{-i\alpha(t'-t)+t'-t-ix(\sinh(t')-\sinh(t))} \\
&= \frac{1}{\pi^2}\int d\eta\int d\xi e^{-(i\alpha-1)\xi-2ix\sinh(\xi)\cosh(\eta)} \\
&= \frac{-2i}{\pi}\int d\eta H^{(2)}_{2i\alpha-2}(-i2x\gamma).
\end{aligned} \tag{S68}$$

Finally, we have only the remaining first two positive terms to evaluate. We will combine these two terms together when evaluating. Hence,

$$\begin{aligned}
|H^{(2)}_{i\alpha-1}(-ix)|^2 + |H^{(2)}_{i\alpha+1}(-ix)|^2 &= \frac{1}{-i\pi}\int dt e^{(i\alpha-1)t-ix\sinh(t)}\frac{1}{i\pi}\int dt' e^{(-i\alpha-1)t'+ix\sinh(t')} \\
&+ \frac{1}{-i\pi}\int dt e^{(i\alpha+1)t-ix\sinh(t)}\frac{1}{i\pi}\int dt' e^{(-i\alpha+1)t'+ix\sinh(t')} \\
&= \frac{1}{\pi^2}\int dt\int dt' e^{i\alpha t-t-ix\sinh(t)-i\alpha t'-t'+ix\sinh(t')} \\
&+ \frac{1}{\pi^2}\int dt\int dt' e^{i\alpha t+t-ix\sinh(t)-i\alpha t'+t'+ix\sinh(t')} \\
&= \frac{1}{\pi^2}\int dt\int dt' e^{i\alpha t-ix\sinh(t)-i\alpha t'+ix\sinh(t')}\left[e^{-t-t'}+e^{t+t'}\right] \\
&= \frac{4}{\pi^2}\int d\eta\int d\xi e^{-2i\alpha\xi-2ix\sinh(\xi)\cosh(\eta)}\cosh(2\eta) \\
&= \frac{-4i}{\pi}\int d\eta H^{(2)}_{2i\alpha}(-i2x\gamma)\cosh(2\eta) \\
&= \frac{-4i}{\pi}\int d\eta\delta H^{(2)}_{2i\alpha}(-i2x\gamma).
\end{aligned} \tag{S69}$$

Note we have recovered the same boost factor as in the Minkowski frame case; $\delta = 2\gamma^2 - 1 = \cosh(2\eta)$. Let us now combine all the derivative terms together to yield,

$$\begin{aligned}
|H^{(2)'}_{i\alpha}(-ix)|^2 &= \frac{1}{4}\int d\eta\left[\frac{-4i}{\pi}\delta H^{(2)}_{2i\alpha}(-i2x\gamma) - \frac{-2i}{\pi}H^{(2)}_{2i\alpha-2}(-i2x\gamma) - \frac{-2i}{\pi}H^{(2)}_{2i\alpha+2}(-i2x\gamma)\right] \\
&= -\frac{i}{\pi}\int d\eta\left[\delta H^{(2)}_{2i\alpha}(-i2x\gamma) - \frac{1}{2}\left(H^{(2)}_{2i\alpha+2}(-i2x\gamma) + H^{(2)}_{2i\alpha-2}(-i2x\gamma)\right)\right]
\end{aligned} \tag{S70}$$

Now that we have all of our pieces together, we find our emission spectrum to be

$$\frac{d\Gamma}{d^2k_\perp} = -\frac{i\alpha}{2\pi a}\int d\eta \sinh\left(\pi\omega_r/a\right)\coth\left(\omega_r/(2T)\right)e^{\frac{\pi\omega_r}{a}} \times \Bigg[\left(\delta + 2\left(\frac{v_0 k_y}{k_\perp}\right)^2\right)H^{(2)}_{2i\omega_r/a}\left(\frac{-2ik_\perp\gamma}{a}\right) - \frac{1}{2}\left(H^{(2)}_{2i\omega_r+2}\left(\frac{-2ik_\perp\gamma}{a}\right) + H^{(2)}_{2i\omega_r-2}\left(\frac{-2ik_\perp\gamma}{a}\right)\right)\Bigg]. \tag{S71}$$

If we are considering a nonrelativistic channeling oscillation, then the term proportional to $\left(\frac{v_0 k_y}{k_\perp}\right)^2$ will be negligible. Moreover, note that we are integrating the above expression over the rapidity η. Restricting our emission spectrum to a window of constant rapidity, i.e. we maintain the assumption of constant acceleration and velocity as in the case applied to the experiment, we then obtain our total emission spectrum in the Rindler frame,

$$\frac{d\Gamma}{d^2k_\perp} = -\frac{i\alpha}{2\pi a}\sinh\left(\pi\omega_r/a\right)\coth\left(\omega_r/(2T)\right)e^{\frac{\pi\omega_r}{a}} \times \left[\delta H^{(2)}_{2i\omega_r/a}\left(\frac{-2ik_\perp\gamma}{a}\right) - \frac{1}{2}\left(H^{(2)}_{2i\omega_r+2}\left(\frac{-2ik_\perp\gamma}{a}\right) + H^{(2)}_{2i\omega_r-2}\left(\frac{-2ik_\perp\gamma}{a}\right)\right)\right]. \tag{S72}$$

In order to confirm that this expression matches the Minkowski expression, we must set the temperature of the background Rindler bath to the FDU temperature, $T = \frac{a}{2\pi}$, and compare. As such the temperature prefactor in the above expression reduces to,

$$\begin{aligned}\sinh\left(\pi\omega_r/a\right)\coth\left(\omega_r/(2T)\right)e^{\frac{\pi\omega_r}{a}} &= \sinh\left(\pi\omega_r/a\right)\coth\left(\pi\omega_r/(a)\right)e^{\frac{\pi\omega_r}{a}} \\ &= \cosh\left(\pi\omega_r/a\right)\, e^{\frac{\pi\omega_r}{a}} \\ &= \frac{1}{2}\left[1 + e^{2\pi\omega_r/a}\right].\end{aligned} \tag{S73}$$

Utilizing the above expression, we find our emission rate in complete agreement with the Minkowski case, Eq. (S51), when we identify the energy gap of the Unruh-DeWitt detector with the Rindler frequency, $\Delta E = \omega_r = k_x A\Omega$. Hence,

$$\frac{d\Gamma}{d^2k_\perp} = -\frac{i\alpha}{4\pi a}\left[\delta H^{(2)}_{2i\omega_r/a}\left(\frac{-2ik_\perp\gamma}{a}\right) - \frac{1}{2}\left(H^{(2)}_{2i\omega_r+2}\left(\frac{-2ik_\perp\gamma}{a}\right) + H^{(2)}_{2i\omega_r-2}\left(\frac{-2ik_\perp\gamma}{a}\right)\right)\right] \times \left[1 + e^{2\pi\omega_r/a}\right]. \tag{S74}$$

H. Acceleration Via Radiation Reaction

Due to the similarity to channeling radiation, let us begin by presenting a simple example and compute the acceleration produced by radiative energy loss, i.e. bremsstrahlung. As such, we have

$$\begin{aligned}\frac{dE}{dx} &= -\frac{E}{x_0} \\ \Rightarrow E(x) &= E_0 e^{-x/x_0}.\end{aligned} \tag{S75}$$

For the sake of clarity, here we have $E = m\gamma = \sqrt{p^2 + m^2}$ and $p = mv\gamma = mu$. The parameter x_0 is the radiation length. Note, radiation lengths are typically on the order of $\sim$ cm, i.e. they are macroscopic quantities. As we will see, this parameter is what ultimately sets the length/time scale for the system and thus the acceleration as well. Let us consider an ultra-relativistic particle and assume the initial energy will be entirely due to the momentum $p_0 = mu_0$, with $u_0 = v_0\gamma_0$ being the initial proper velocity. Thus, the expression for radiative energy loss will reduce to the change in the relativistic momentum as a function of distance in the lab frame,

$$p(x) = p_0 e^{-x/x_0}. \tag{S76}$$

Note that this quantity determines the change in momentum of the positron. It is this change in momentum that gives rise to a force, and thus acceleration. Keeping all aspects of the analysis in the lab frame, let us recall the definition of force is the change in momentum per unit time, $f = \frac{dp}{dt}$. Thus

$$\frac{dp}{dt} = ma'$$
$$\Rightarrow a' = \frac{1}{m}\frac{dp}{dt}. \tag{S77}$$

Here, we explicitly have the proper acceleration, a', from the relativistic version of Newtons law $f = ma_{lab}\gamma^3$. Again in the ultra relativistic limit, we obtain

$$a' = \frac{1}{mx_0}\frac{dx}{dt}p_0 e^{-x/x_0}$$
$$a'_0 = \frac{\gamma_0}{x_0}. \tag{S78}$$

Note that here we made use of the fact that $\frac{dx}{dt} = v = 1$ for relativistic velocities. Also, the approximation $e^{-x/x_0} = .96 \sim 1$ is equivalent to the time independence of the acceleration for the system. Note, using the parameters of the experiment; $\gamma = 3.5 \times 10^5$ and $x_0 = 9.37$ cm for silicon, the proper acceleration is given by $a'_0 = 74$ ceV. Although this acceleration scale is relatively large in the Unruh setting, we need to find an acceleration scale at or beyond ~ 100 GeV, based on the energy scales in the experiment. Note again, the acceleration time scale is set by $t = x_0/c$ and therefore reflects the bulk acceleration produced by radiative energy loss. In order to bring about a larger acceleration we must look at the photon emission microscopically so as to obtain a more accurate acceleration time scale for each process individually.

From the power spectrum of the actual data set, we have a max photon frequency of $\omega_0 = 150$ GeV. Note, this was also the first photon frequency to thermalize. Let us then examine the acceleration produced via this emission. Using this frequency as the change in momentum, $|\Delta p| = |k| = \omega_0$. This momentum change occurs during the lifetime of the emission process. Taking the physical size of the photon to be $\Delta x = \lambda/2$, we can then determine the emission time to be, $\Delta t = \Delta x/c = \frac{\pi}{\omega_0}$. Then, using $\frac{\Delta p}{\Delta t} = ma'$, our proper acceleration is given by,

$$a' = \frac{\omega_0^2}{\pi m} \tag{S79}$$

This is a proper acceleration but it is written in terms of the lab frequency. What is important to note is that when written in terms of proper quantities, the proper acceleration, $a' = \frac{\omega_0'^2\gamma^2}{\pi m}$, boosts as γ^2. More importantly, when computing the FDU temperature for the emission, we find a recoil/radiation reaction temperature, T_{RR}. Hence,

$$T_{RR} = \frac{\omega_0'^2\gamma^2}{2m\pi^2}. \tag{S80}$$

Note, we now have an FDU temperature which depends explicitly on the recoil kinetic energy, $\omega^2/2m$ which is imparted on the positron by the emission. It is this acceleration, produced by the radiation reaction itself, that we need to look at in order to obtain large enough accelerations. Lets examine the temperature produced by the recoil of the maximum frequency in the data set, $\omega_0 \sim 150$ GeV. As such, we expect to find a temperature of,

$$T_{RR} = 2.23\ PeV \tag{S81}$$

I. Bekenstein-Hawking Area-Entropy

In consideration of the first law of thermodynamics, $d\mathcal{E} = TdS$, we can use this relation to determine the amount of entropy via the photon emission, here $\mathcal{E}$ is the energy radiated away in the proper frame. Note, all quantities here are necessarily in the proper frame. Recalling first that the temperature is given by $T_{RR} = \frac{\omega_0'^2\gamma^2}{2m\pi^2}$. From here we find that the temperature can be written in terms of the energy via $T_{RR} = \frac{\omega_0'^2 E^2}{2m^3\pi^2}$ if we recall that $\gamma = E/m$. Note we express all our proper quantities in terms of the lab energy E. Then, recalling the proper energy is related to the lab energy via $d\mathcal{E} = dE/\gamma$, we have

$$\begin{aligned} dS &= \frac{2m\pi^2}{\omega_0'^2\gamma^2} d\mathcal{E} \\ &= \frac{2m^4\pi^2}{\omega_0'^2}\frac{dE}{E^3} \\ \Rightarrow S &= \frac{c^8}{\hbar^2}\frac{m^4\pi^2}{\omega_0'^2}\frac{1}{E^2}. \end{aligned} \tag{S82}$$

Here, we assumed the initial entropy is zero. Note this expression is consistent with black hole entropy, and the Bekenstein-Hawking area-entropy law, since we have $ST = \frac{m}{2}$. We will now utilize this expression to examine the difference between the initial and final state entropy. Writing it explicitly in terms of the initial and final energy, in order to compare it to the change in the horizon area, we find,

$$\begin{aligned} \Delta S &= \frac{c^8}{\hbar^2}\frac{m^4\pi^2}{\omega_0'^2}\left[\frac{1}{E_f^2} - \frac{1}{E_i^2}\right] \\ &= \frac{c^8}{\hbar^2}\frac{m^4\pi^2}{\omega_0'^2}\left[\frac{1}{(E_i-\Delta E)^2} - \frac{1}{E_i^2}\right]. \end{aligned} \tag{S83}$$

It is this entropy that we will compare with the change in Rindler horizon area in order to experimentally confirm the proportionality factor of $\frac{1}{4}$. Note, $E_f = E_i - \Delta E$, with ΔE given by the energy radiated by the positron and is determined by the integral of the power spectrum over frequency and time. It is also this energy, when boosted into the proper frame, that we will use to determine the change in the area of the Rindler horizon. As such, the change in area, ΔA, generated by a flux of energy momentum across a Rindler horizon is given by [30],

$$\Delta A = 8\pi G \int d^2y \int_0^\infty dv v T^{\mu\nu} k_\mu k_\nu. \tag{S84}$$

Here, $y = (y_1, y_2)$, is the transverse area, $v = \frac{x+t}{2}$ determines the spacetime propagation of the light like vector $k^\mu = (1,1,0,0)$ which characterizes the light rays which span the horizon, and $T^{\mu\nu}$ is the energy momentum ten sor of the matter/energy that crosses through the horizon. The total mass/energy emitted by the positron into the horizon will give us $T^{\mu\nu}k_\mu k_\nu = \frac{\Delta E}{\gamma c^2}\delta(v-v_0)\delta^2(y)$. Here, $\frac{\Delta E}{\gamma}$ is the total mass/energy emitted by the positron in the proper frame. If we assume the energy is emitted at $x = \frac{1}{a}$ and $t = 0$, then this implies the positron is at zero velocity in the proper frame. The mass/energy will then cross the horizon at $x = \frac{1}{a}$ and $t = \frac{1}{a}$. As such we have, $v_0 = \frac{1}{a}$. Therefore, the total change in the horizon area is given by

$$\begin{aligned} \Delta A &= \frac{8\pi G \Delta E}{\gamma a c^2} \\ &= \frac{Gc^5}{\hbar}\frac{8\pi^2 m^4 \Delta E}{E^3\omega_0'^2} \end{aligned} \tag{S85}$$

The Bekenstein-Hawking area-entropy law states that $\Delta A/\Delta S = 4\ell_p^2$, where ℓ_p^2 is the Planck area. Writing the ratio of the area change to the entropy change yields,

$$\frac{\Delta A}{\Delta S} = \ell_p^2 \frac{8\Delta E}{E_i^3}\left[\frac{1}{(E_i-\Delta E)^2} - \frac{1}{E_i^2}\right]^{-1}. \tag{S86}$$

Note, to lowest order in acceleration and energy change, the area-entropy ratio is indeed trivially satisfied. This is due to the fact that the entropy is obtained by direct integration of the 1st law, $dE = TdS$, which a priori satisfies the Bekenstein-Hawking condition $S = A/4$. When expanding the entropy to first order in $\Delta E = 0$, and forming the ratio, this yields the first order term which will always be satisfied. To see this, let us consider more terms in the expansion. Hence,

$$\begin{aligned} \frac{\Delta A}{\Delta S} &= \ell_p^2 \frac{8\Delta E}{E_i^3}\left[\frac{1}{(E_i-\Delta E)^2} - \frac{1}{E_i^2}\right]^{-1} \\ &\sim 4\ell_p^2\left[1 - \frac{3\Delta E}{2E_i} + \frac{\Delta E^2}{4E_i^2} + \frac{\Delta E^3}{8E_i^3} + \cdots\right]. \end{aligned}$$

What we find is that the zeroth order term is always satisfied. However, what is required to confirm the presence of thermality is that all terms satisfy the relation $S = A/4$, even with $\Delta E \neq 0$. Conceptually what this means is that by the original integration of the 1st law, we fix $S_i = \frac{A_i}{4}$. This is the zeroth order "initial condition" of the integration. Then, we must have ΔE evolve in such a way that the change in the area and entropy also obey $\Delta S = \frac{\Delta A}{4}$. This is not always the case in fact. The ΔE that presents itself in the above expressions must come from a thermalized observable. In other words, if the integral of ΔE came from a spectra with, e.g., a particle resonance present, then the resultant ΔE will not satisfy the Bekenstein-Hawking relationship.

To better illustrate this point, we have deformed the power spectrum data set by adding in a Gaussian resonance. The was accomplished by scaling the y-component of each data point as follows;

$$data_y[i] \rightarrow data_y[i] * \left(1 + \frac{\alpha}{\sigma\sqrt{2\pi}} e^{(-1/2((i-\mu)/\sigma)^2}\right). \tag{S87}$$

For our purpose we used $\alpha = 30$, $\sigma = 7$ and $\mu = 50\ GeV$ and $\mu = 70\ GeV$. All errors were kept the same. We then use this spectrum in the integral of ΔE. The resultant area-entropy ratio diverges from the Bekenstein-Hawking relationship and therefore reflects the lack of thermality in the system, see figure 1 below for an illustration of this phenomena.

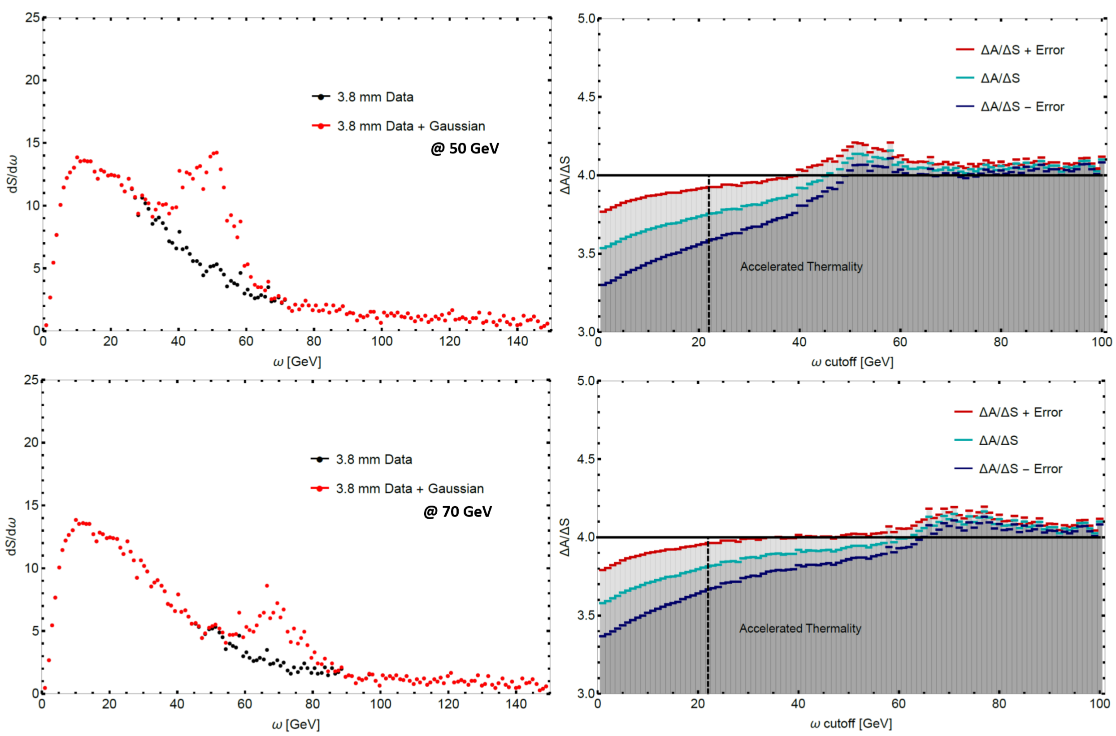

FIG. S1. **Area-entropy ratio for a non-thermal power spectrum:** By including a Gaussian resonance in the power spectrum data (LEFT), we can examine its effect on the area-entropy ratio (RIGHT). Note the Bekenstein-Hawking relationship fails for non-thermal power spectra simulated by including Gaussian resonances at 50 GeV (TOP) and 70 GeV (BOTTOM).

What the above figures demonstrate is that for the full dynamics over the integration of ΔE, the area-entropy ratio will not be thermal if the observable analyzed is not thermal. In other words, satisfying the Bekenstein-Hawking relationship $S = A/4$ must happen at all orders in ΔE and not just the initial condition, i.e. the initial condition satisfies $S_i = A_i/4$ and this relationship is upheld throughout the integration such that $\Delta S = \Delta A/4$, for all ΔE of a thermal system.

J. Notes on the Energy Gap

For the sake of simplicity, let us examine the radiation emission/absorption of a field $\phi(x)$ from a second quantized field $\psi(x)$, i.e. not an Unruh-DeWitt detector, in the Rindler frame. The integration over the spatial modes yields delta functions whose argument encodes conservation of momentum, e.g. $\delta(p_f - p_i + k)$, via

$$\mathcal{A}_{i\to f} \sim \int d^3x\sqrt{-g}\psi_f^*(x,p)\psi_f(x,p)\phi(x,k)$$
$$\sim \delta(p_f - p_i + k).$$

The expression for momentum conservation, in the case of the channeling experiment, can be rather complicated on account of there being a channeling oscillation, recoil, and all other processes present. When looking at the change in the electron Rindler energy $\Delta E = E_f - E_i$, we simply Taylor expand about the photon frequency. We must also note, that in an inertial comoving frame, the electron Rindler energy coincides with the electron Minkowski energy. Then, as an example, for the above conservation of momentum statement (taking $p_i = 0$ for the sake of simplicity), we will have

$$\begin{aligned}\Delta E &= \sqrt{(p_f)^2 + m^2} - E_i \\ &= \sqrt{(-k)^2 + m^2} - m \\ &= \sqrt{(\omega)^2 + m^2} - m \\ &\sim \frac{\omega^2}{2m}\end{aligned} \tag{S88}$$

We also expect there to be a pure channeling frequency Ω term based on both the Cozzella analysis [23] as well as the fact that the same term is present in the energy gap of an Unruh-DeWitt detector for the anomalous doppler effect. In our Rindler analysis, this term was most likely thrown away when we took the electron current to be at constant transverse velocity. However, by keeping the oscillation in the phase, we obtain the $\Delta E = \omega_r = k_x A\Omega$. For a dipole oscillator, when heavily boosted, this becomes beamed yielding $k_x \sim \omega$, in the lab frame. This yields our linear term, $A\Omega\omega$. These three terms together comprise the fiducial terms of our energy gap,

$$\Delta E \sim \Omega + \Omega A\omega + \frac{\omega^2}{2m}. \tag{S89}$$

Finally, since we do not know what the exact dispersion relation is in the channeling experiment, we employed a more general polynomial in the emitted photon's frequency, i.e. $\Delta E = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3$. We include one more term beyond the known ones for the sake of completeness. To match the calculated spectrum to the data we also include an over all scaling factor s for the spectrum and parameter, $\tilde{a}$, to fit the acceleration. The measured best fit parameters are presented below.

Energy	χ^2/ν	s	$\tilde{a}$ [PeV]	a_0 [GeV]	a_1	a_2 [GeV^{-1}]	a_3 [GeV^{-2}]
30 GeV	1.114	12.93	7.939	-0.00197	0.0120	-846.2	0.4691

TABLE S1. The best fit parameters for our theoretical power spectrum with the energy gap for the 3.8 mm channeling crystal sample at the energy where the chi-squared statistic first meets the 1 standard deviation criterion. Our reduced chi-squared statistics shows that the data can be rigorously fit by the power spectrum with the energy gap thermalized at the FDU temperature.

K. Notes on the Thermalization Time

Lets go back and consider an Unruh-DeWitt detector coupled to a massless scalar field. To begin, let us recall the response function, i.e. transition rate for the Unruh-DeWitt detector [8], is given by,

$$\Gamma = q^2 \int d\xi e^{-i\Delta E\xi} G^{\pm}[x', x]. \tag{S90}$$

We should note here that the Wightman function $G^{\pm}[x', x]$ of the emitted scalar field, when evaluated on a hyperbolic trajectory, is given by $G = -\frac{1}{(2\pi)^2}\frac{a^2}{\sinh^2(a\tau/2)}$. The resultant integration yields the standard thermal response function associated with the Unruh effect,

$$\Gamma = q^2 \frac{\Delta E}{2\pi}\frac{1}{e^{2\pi\Delta E/a} - 1}. \tag{S91}$$

The inverse of this expression is the thermalization time. However, we must note that the above expression utilized the Wightman function $G \sim 1/\sinh^2(a\tau/2)$, which had already integrated out the scalar fields frequency. To see this, let us examine the computation of the Wightman function explicitly,

$$\begin{aligned}
G^{\pm}[x', x] &= \langle 0| \hat{\phi}(x')\hat{\phi}(x) |0\rangle \\
&= \frac{1}{2(2\pi)^3}\iint \frac{d^3k' d^3k}{\sqrt{\omega'\omega}} \langle 0| \left[\hat{a}_{\mathbf{k}'} e^{i(\mathbf{k}'\cdot\mathbf{x}' - \omega' t')} + \hat{a}^{\dagger}_{\mathbf{k}'} e^{-i(\mathbf{k}'\cdot\mathbf{x}' - \omega' t')}\right] \left[\hat{a}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} + \hat{a}^{\dagger}_{\mathbf{k}} e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega t)}\right] |0\rangle \\
&= \frac{1}{2(2\pi)^3}\iint \frac{d^3k' d^3k}{\sqrt{\omega'\omega}} \langle 0| \hat{a}_{\mathbf{k}'}\hat{a}^{\dagger}_{\mathbf{k}} e^{i(\mathbf{k}'\cdot\mathbf{x}' - \mathbf{k}\cdot\mathbf{x} - \omega' t' + \omega t)} |0\rangle \\
&= \frac{1}{2(2\pi)^3}\iint \frac{d^3k' d^3k}{\sqrt{\omega'\omega}} e^{i(\mathbf{k}'\cdot\mathbf{x}' - \mathbf{k}\cdot\mathbf{x} - \omega' t' + \omega t)} \delta(\mathbf{k}' - \mathbf{k}) \\
&= \frac{1}{2(2\pi)^3}\int \frac{d^3k}{\omega} e^{i(\mathbf{k}\cdot\Delta\mathbf{x} - \omega\Delta t)}.
\end{aligned} \tag{S92}$$

Note, the frequency here is *precisely* the frequency of the emitted particle; in this case a massless scalar but in our manuscript it is the photon. Integration of this expression over the frequencies yields the typical $G \sim 1/\sinh^2(a\tau/2)$ form and everything reduces to the standard Unruh case. Utilizing the unitegrated expression, the response function, and thus the basis for the power spectrum derived in the manuscript is given by,

$$\Gamma = \frac{q^2}{2(2\pi)^3}\int d\xi \int \frac{d^3k}{\omega} e^{-i\Delta E\xi} e^{i(\mathbf{k}\cdot\Delta\mathbf{x} - \omega\Delta t)}. \tag{S93}$$

Letting $\Delta x = 0$, which is the "non relativistic" approximation used in the manuscript that successfully yielded the Larmor formula, and recalling $\Delta t = \frac{2}{a}\sinh(a\xi/2)\gamma$ [9], we then have

$$\Gamma = \frac{q^2}{2(2\pi)^3}\int d\xi \int \frac{d^3k}{\omega} e^{-i\Delta E\xi} e^{-i\omega\frac{2}{a}\sinh(a\xi/2)\gamma}. \tag{S94}$$

Again, making the change of variables $w = a\xi/2$ and recalling the Hankel identity Eqn. (S40), we then have

$$\begin{aligned}
\Gamma &= \frac{q^2}{2(2\pi)^3}\frac{2}{a}\int dw \int \frac{d^3k}{\omega} e^{-i\Delta E\xi} e^{-i\omega\frac{2}{a}\sinh(a\xi/2)\gamma} \\
&= \frac{q^2}{(2\pi)^2}\frac{2}{a}\int dw \int d\omega\omega e^{-i\frac{2\Delta E}{a}w - i\omega\frac{2}{a}\sinh(w)\gamma} \\
&= -i\alpha\frac{2}{a}\int d\omega\omega H^{(2)}_{\frac{2i\Delta E}{a}}\left(-i\frac{2\omega\gamma}{a}\right).
\end{aligned} \tag{S95}$$

This is, of course, the analogous expression for the AQED emission rate, Eqn. (S41) along with $\frac{d\mathcal{S}}{d\omega} = \frac{d\Gamma}{d\omega}\omega$, but for the massless scalar field rather than photon; the main difference being the two additional Hankel functions which come from the polarization of the photon. Here, we see explicitly why we still have to integrate over the frequency. It is because we must start with the Wightman function which has not integrated the frequency out. This also enables us to include higher order frequency terms in the energy gap of the Unruh-DeWitt detector in a self consistent way. This essentially means that we have switched the order of integration between the proper time of the detector and the emitted photons frequency.

In terms of how to interpret the thermalization time, we must first comment on the above expressions relationship to the standard Unruh response function. Modulo the approximation $\Delta x = 0$, the integration over the frequency should reduce to the appropriate approximation of the standard Unruh form ;

$$\begin{aligned} \Gamma &= -i\alpha \frac{2}{a} \int d\omega \omega H^{(2)}_{\frac{2i\Delta E}{a}} \left(-i \frac{2\omega\gamma}{a} \right) \\ &\sim q^2 \frac{\Delta E}{2\pi} \frac{1}{e^{2\pi\Delta E/a} - 1}. \end{aligned} \tag{S96}$$

This, of course, is valid provided switching the proper time and frequency integrals is valid. This quantity is to be interpreted as a decay/excitation rate in the conventional sense. In other words, given an initial population N_0 of excited detectors, the population as a function of time will be $N(t) = N_0 e^{-\Gamma t}$. The thermalization time is given by $\tau = 1/\Gamma$ and determines the time until the population has been reduced by $1/e$. In the case of particle decays, this is known as the particle lifetime. This is also the quantity that is used as the time necessary for thermality to take hold in the Unruh setting [11]. Now, we note that the thermalization time is simply the unintegrated form,

$$\tau = \frac{1}{\int_0^\infty \frac{d\Gamma}{d\omega} d\omega}. \tag{S97}$$

So the question is now, how does all of this change with a frequency dependent energy gap? The easiest way to see this is to look to an example of a particle decay with 3 branching ratios. Each process has a rate Γ_1, Γ_2, and Γ_3. The total decay rate is the sum off all three terms $\Gamma_{tot} = \sum \Gamma_i$. It is this total decay rate which shows up in the exponential decay, $N(t) = N_0 e^{-\Gamma_{tot} t}$. The decay lifetime, or thermalization time, is the reciprocal of the sum, $\tau = 1/\sum \Gamma_i$, and not the sum of reciprocals, $\tau \neq 1/\Gamma_1 + 1/\Gamma_2 + 1/\Gamma_3$. Most importantly, each decay pathway has its own decay rate and will decay according to its own lifetime. The total decay rate simply combines all contributions for an ensemble system. The example here demonstrates what happens with discrete decay pathways.

What about having an infinite number of decay/excitation pathways? Consider a “particle” with an infinite number of Unruh-DeWitt detectors and a continuous distribution of energy gaps. With no degeneracies, we will have precisely one Unruh-DeWitt detector for every single frequency in the electromagnetic spectrum. This means the total decay rate will be summing over each mode $\Gamma_t = \sum_{i=0}^{\infty} \Gamma(id\omega)$. Since we have infinitesimal differences between our energy gaps, $d\omega$, then we should likewise have infinitesimal differences in our excitation rate, $d\Gamma(id\omega) = \Gamma((i+1)d\omega) - \Gamma(id\omega)$. Starting from the excitation rate of the zero energy gap, Γ_0, our sum then becomes $\Gamma_t = \Gamma_0 + \sum_{i=0}^{\infty} d\Gamma(id\omega)$. For excitations, a zero energy gap will never excite and the rate is identically zero, $\Gamma_0 = 0$. In the sum, we can multiply and divide by $d\omega$ to convert our expression into a Riemann sum, $\Gamma_t = \sum_{i=0}^{\infty} \frac{d\Gamma(id\omega)}{d\omega} d\omega$. This, of course, can now be transformed into an integral over the photon frequency and completes the derivation of the standard excitation rate for a continuous system as an integral over all internal energy gaps,

$$\Gamma_t = \int_0^\infty \frac{d\Gamma(\omega)}{d\omega} d\omega. \tag{S98}$$

Let us now turn to the question of the thermalization time of an individual frequency. Recalling from the discrete case that the total decay rate was the sum of each rate of the individual decay modes we must also point at that each decay mode stands on its own. In other words, each individual decay pathway obeys its own decay rate, but the total decay rate depends on the sum. In just the same way, the individual excitation rate of a particular mode in a continuous system will also stand alone, and thermalize at its own prescribed time in the Unruh picture. In order to determine the excitation rate of an individual mode, ω' we must find the excitation rate of that frequency, i.e. $\Gamma(\omega')$, we must integrate up to that frequency in our total excitation rate. Thus,

$$\begin{aligned} \int_0^{\omega'} \frac{d\Gamma(\omega)}{d\omega} d\omega &= \Gamma(\omega') - \Gamma(0) \\ &= \Gamma(\omega') \end{aligned} \tag{S99}$$

Here we made use of the fundamental theorem of calculus and the fact that the excitation rate of a zero energy gap is zero. What this shows is that as we integrate over the frequency of a continuous detector energy gap, each frequency, ω', has an excitation rate, $\Gamma(\omega') = \int_{i=0}^{\omega'} \frac{d\Gamma(\omega)}{d\omega} d\omega$. The integral up to ω' gives precisely the individual excitation rate in the sum, of the total excitation rate, that characterizes this specific process individually. This process will also thermalize at its own time $\tau' = 1/\Gamma(\omega')$. This is precisely the scenario we saw in the discrete case. Each process thermalizes at its own time but the total thermalization time (meaning all processes have thermalized) is the inverse of the total rate.

L. Notes on the Semiclassical Vector Current

Let us go back and "derive" the Unruh-DeWitt detector from the QED current interaction. This will show how it is possible to incorporate particle interactions into a two level system in a self consistent way. For the photon field, $\hat{A}^\mu(x)$, We will have the following action [33],

$$\hat{S}_I = q \int d^4x \hat{\bar{\psi}} \gamma_\mu \hat{\psi} \hat{A}^\mu(x). \tag{S100}$$

Here we have the Dirac current, $\hat{j}_\mu = \hat{\bar{\psi}} \gamma_\mu \hat{\psi}$, which we would like to model as a semi-classical vector current. To see how this is accomplished we begin by recalling that for spinors $u(s,p)$ and $v(s,p)$ of spin s and momentum p that are created by $\hat{a}^\dagger_{s,p}$ and $\hat{b}^\dagger_{s,p}$, we have the following electron field operators

$$\begin{aligned} \hat{\psi}(x,t) &= \int d^3p \sum_s \left[\hat{a}_{s,p} u(s,p) \phi_p(x,t) + \hat{b}^\dagger_{s,p} v(s,p) \chi_p(x,t) \right], \\ \hat{\bar{\psi}}(x,t) &= \int d^3p \sum_s \left[\hat{a}^\dagger_{s,p} \bar{u}(s,p) \phi^*_p(x,t) + \hat{b}_{s,p} \bar{v}(s,p) \chi^*_p(x,t) \right]. \end{aligned} \tag{S101}$$

The positive and negative frequency modes are given by $\phi_p(x,t)$ and $\chi_p(x,t)$ respectively. Normally these modes are plane waves, however in more general spacetimes our only requirement is that they are positive and negative frequency modes with respect to the particle's/detector's proper time. Using these fields, we will formulate the Dirac current, $\hat{j}_\mu = \hat{\bar{\psi}} \gamma_\mu \hat{\psi}$. Let us now consider the transition element between initial, $|E_i\rangle$, and final, $|E_f\rangle$, electron energy state. We are also neglecting any spin effects. Now, focusing strictly on electrons, i.e. no antiparticles, the only surviving element of the current will be given by,

$$\hat{j}_\mu(x) = \bar{u}(p_f) \gamma_\mu u(p_i) \phi^*_{E_f}(x,t) \hat{a}^\dagger_{p_f} \hat{a}_p \phi_{E_i}(x,t). \tag{S102}$$

In the above expression, we still have the spinor degrees of freedom to deal with. For this, we will make use of the Gordon identity,

$$\bar{u}(p_f) \gamma^\mu u(p_i) = \bar{u}(p_f) \left[\frac{p^\mu_f + p^\mu_i + i\sigma^{\mu\nu}(p_{f\nu} - p_{i\nu})}{2m} \right] u(p_i). \tag{S103}$$

Neglecting the spin coupling yields the kinematic term,

$$\bar{u}(p_f) \gamma^\mu u(p_i) = \bar{u}(p_f) \left[\frac{p^\mu_f + p^\mu_i}{2m} \right] u(p_i). \tag{S104}$$

Finally, we will make the assumption that at the level of the semiclassical vector current, that the momentum remains constant throughout the radiative process. As such, we have

$$\bar{u}(p_f) \gamma_\mu u(p_i) = u_\mu. \tag{S105}$$

We now make use of the fact that our positive and negative frequency mode solutions can be separated into their spatial and temporal components via $\phi(x,\tau) = g(x)e^{-iE\tau}$. We have chosen to parametrize our fields via the electron's proper time, τ, to incorporate the Rindler coordinate chart when analyzing the accelerated case. Our current now reduces to

$$\hat{j}_\mu(x) = u_\mu g_f^*(x) g_i(x) e^{iE_f\tau} \hat{a}^\dagger_{p_f} \hat{a}_{p_i} e^{-iE_i\tau}. \tag{S106}$$

We note that for sufficiently localized electronic wave functions, e.g. with a wavelength much smaller than the wavelength of emitted radiation, we have $g_f^*(x)g_i(x) = \delta^3(x - x_{tr})$ along the classical trajectory of the electron; which is assumed to be uniform. Finally, by attaching the time-dependence to the creation and annihilation operators we have

$$e^{iE_f\tau} \hat{a}^\dagger_{p_f} \hat{a}_{p_i} e^{-iE_i\tau} = e^{i\hat{H}\tau} \hat{m}(0) e^{-i\hat{H}\tau} = \hat{m}(\tau). \tag{S107}$$

Here we defined the Heisenberg evolved monopole moment operator $\hat{m} = e^{i\hat{H}\tau}\hat{m}(0)e^{-i\hat{H}\tau}$ where $\hat{m}(0)$ is defined as $\hat{m}(0)\,|E_i\rangle = |E_f\rangle$ with E_i and E_f the initial energy and final energy of the electron moving along the trajectory, x_{tr}, of the current. The energy gap of the detector is $\Delta E = E_f - E_i$ and the system is normalized via $1 = |\,\langle E_f|\,\hat{m}(0)\,|E_i\rangle\,|$. As such, we have transformed our fermionic current into a semi-classical charged current coupled to an Unruh-DeWitt detector,

$$\hat{j}_\mu(x) = u_\mu \hat{m}(\tau)\delta^3(x - x_{tr}). \tag{S108}$$

We have provided a proof that the Dirac current can be reduced to a semiclassial vector current coupled to an Unruh-DeWitt detector. The energy gap is defined by the difference between the initial and final electron energy with respect to the proper time of the charged particle. Note, that the energy gap of our Unruh-DeWitt detector is given by, $\Delta E = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3$. Since the monopole moment operator creates and/or annihilates states with definite momentum then we need to think of our semiclassical vector current coupled to a continuum of Unruh-DeWitt detectors; one for each frequency of our energy gap.

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